Functional Analysis Exam Solutions

Question 1

(i) A normed vector space V is a Banach space if it is complete, that is to say every Cauchy sequence in V converges to a point in V.

(ii) Let (x_n) be a Cauchy sequence in V. Write

$$x_n = (\alpha_n^{(1)}, \ldots, \alpha_n^{(k)}).$$

Then

$$\|x_m - x_n\|^4 = |\alpha_m^{(1)} - \alpha_n^{(1)}|^4 + \cdots + |\alpha_m^{(k)} - \alpha_n^{(k)}|^4$$

so for each i

$$|\alpha_m^{(i)} - \alpha_n^{(i)}| \leq \|x_m - x_n\|$$

It follows that the sequence of real numbers, $(\alpha_n^{(i)})$ is Cauchy. The real numbers are complete, so we have $\alpha^{(i)} \in \mathbb{R}$ such that $|\alpha_n^{(i)} - \alpha^{(i)}| \to 0$ as $n \to \infty$.

By the algebra of limits

$$|\alpha_n^{(1)} - \alpha^{(1)}|^4 + \cdots + |\alpha_n^{(k)} - \alpha^{(k)}|^4 \to 0$$

as $n \to \infty$, so $\|x_n - x\| \to 0$ as $n \to \infty$, where

$$x = (\alpha^{(1)}, \ldots, \alpha^{(k)}).$$

Thus the Cauchy sequence (x_n) converges, and V is complete.

(iii) A linear map $T \colon V \to W$ is a bounded linear operator if there is a constant $M \geq 0$ such that $\|Tv\| \leq M\|v\|$ for all $v \in V$.

(iv) Let $T\colon V \to W$ be a bounded linear map. Then we have $M > 0$ such that $\|Tv\| \leq M\|v\|$ whenever $v \in V$.

Let $v_0 \in V$ and $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{M}$. Then if $\|v - v_0\| < \delta$, we have

$$\|Tv - Tv_0\| \leq M\|v - v_0\| < M\delta = \varepsilon$$

It follows that T is continuous.

Conversely, let $T\colon V \to W$ be a continuous map. Then T is continuous at 0. If we take $\varepsilon = 1$ in the definition, we get $\delta > 0$ such that if $\|x\| < \delta$, then $\|Tx\| < 1$.

For any $x \in V$, let

$$v = \frac{\delta x}{2\|x\|}.$$

Then $\|v\| < \delta$, so $\|Tv\| < 1$, and rearranging

$$\|Tx\| \leq \frac{2}{\delta}\|x\|$$

so T is a bounded linear map.

(v) Let $\{e_1, \ldots, e_k\}$ be the standard basis for V. Then

$$\|T(\alpha_1, \ldots, \alpha_k) = |\alpha_1|\|Te_1\| + \cdots + |\alpha_k|\|Te_k\| \leq M\|(\alpha_1, \ldots, \alpha_k)\|$$

where $M = \max(\|Te_1\|, \ldots, \|Te_k\|)$.

Thus T is a bounded linear map, and therefore, by the above, continuous.

Question 2

(i) Certainly $U \subseteq \overline{U}$, so $\overline{U}^{\perp} \subseteq U^{\perp}$.

Let $x \in U^{\perp}$, and $y \in \overline{U}$. Then there is a sequence (y_n) in U with limit y, and $\langle x, y_n \rangle = 0$ for all n. By continuity of the inner product, $\langle x, y \rangle = 0$, so $x \in \overline{U}^{\perp}$. We conclude $\overline{U}^{\perp} = U^{\perp}$.

Let $x \in H$ be the limit of a sequence (x_n) in $U^{\perp}$. Let $y \in U$. Then $\langle x_n, y \rangle = 0$ for all n, and taking limits $\langle x, y \rangle = 0$. Thus $x \in U^{\perp}$, and we conclude $U^{\perp}$ is closed.

(ii) Since f is a continuous map, $\ker f$ is closed. If $f = 0$, the result is trivial. Otherwise, we can find $u \in H$ such that $f(u) = 1$.

Write $H = (\ker f) \oplus (\ker f)^{\perp}$.

$$u = v + w \qquad v \in \ker f,\ w \in (\ker f)^{\perp}.$$

Observe that $f(u) = f(w)$, so $f(w) = 1$. More generally, let $x \in H$. Then

$$f(x - f(x)w) = f(x) - f(x)f(w) = 0$$

so $x - f(x)w \in \ker f$. It follows that $\langle w, x - f(x)w \rangle = 0$, so

$$\langle w, x \rangle = f(x)\langle w, w \rangle.$$

Set $R_f = w/\|w\|^2$. Then by the above

$$\langle R_f, x \rangle = f(x)$$

and we have established existence of the vector R_f.

Let $R'_f \in H$ be another vector where $f(x) = \langle R'_f, x \rangle$ for all $x \in H$. Let $x = R_f - R'_f$. Then

$$\langle x, x \rangle = \langle R_f, x \rangle - \langle R'_f, x \rangle = 0.$$

Therefore $x = 0$, which tells us that $R_f = R'_f$ and establishes uniqueness.

(iii) Certainly

$$R_f = (1, 0, 0, 0, \ldots) \qquad R_g = (0, 0, 0, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots)$$

By conjugate linearity of the inner product

$$R_{f+ig} = R_f - iR_g = (1, 0, 0, -\frac{i}{4}, -\frac{i}{5}, -\frac{i}{6}, \ldots)$$

(iv) Let $U = H$, and T be the zero map. Then $T[U^\perp] = T[\{0\}] = \{0\}$, but $T[U]^\perp = \{0\}^\perp = H$. So the equation is not true in general.

Question 3

(i) Let P be a partially ordered set. Suppose that every totally ordered subset has an upper bound. Then the set P has a maximal element.

(ii) We call a subset $M \subset H$ an *orthonormal basis* if each element of M has norm 1, and $\langle x, y \rangle = 0$ whenever $x, y \in M$ and $x \neq y$, and $\overline{\text{Span}(M)} = H$.

Let H be a Hilbert space. Let $\mathcal{P}$ be the collection of all orthonormal subsets of H, partially ordered by inclusion. Let C be a chain in $\mathcal{P}$. Let

$$A = \bigcup_{B \in C} B.$$

We claim that A is orthonormal. To see this, let $x, y \in A$. Suppose $x \in B_1$ and $y \in B_2$ where $B_1, B_2 \in C$. Then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. But the sets B_1 and B_2 are both orthonormal, so $\langle x, y \rangle = 0$.

Certainly, $\|x\| = 1$ for all $x \in A$. Thus the set A is orthonormal, and $A \in \mathcal{P}$. Certainly $B \subseteq A$ for all $B \in C$. Thus A is an upper bound for the chain C.

By Zorn's lemma, it follows that the set $\mathcal{P}$ has a maximal element, meaning we have an orthonormal set S such that if S is a subset of an orthonormal set A, then $S = A$.

We claim now that $\overline{\text{Span}(S)} = H$, meaning that S is an orthonormal basis.

Suppose otherwise. Then $\overline{\text{Span}(S)} \neq H$, which means $\text{Span}(S)^{\perp} \neq \{0\}$. Pick $x \in \text{Span}(S)^{\perp}$ such that $\|x\| = 1$. Set $A = S \cup \{x\}$. Then A is an orthonormal set containing S.

But this statement contradicts the above, and so $\overline{\text{Span}(S)} = H$ as required.

(iii) We define the *graph* of T to be the set

$$\mathrm{Graph}(T) = \{(T, T(x)) \mid x \in V\} \subseteq V \oplus W.$$

Let V and W be Banach spaces, and let $T\colon V \to W$ be a linear map where the graph is a closed subset of $V \oplus W$. Then T is continuous.

(iv) let $x = (v, w) \in \overline{\mathrm{Graph}(g)}$. Then we have a sequence, (x_n), in $\mathrm{Graph}(g)$ converging to x. Write $x_n = (v_n, g(v_n))$ where $v_n \in H$. Then $\|v_n - v\| \to 0$ as $n \to \infty$.

Fix $u \in H$. Then by continuity of the inner product, $\langle f(u), v_n - v\rangle \to 0$ as $n \to \infty$. But

$$\langle f(u), v_n - v\rangle = \langle u, g(v_n) - g(v)\rangle \to 0$$

as $n \to \infty$. But we know that $\|g(v_n) - w\| \to 0$ as $n \to \infty$. Hence $w = g(v)$, and so $x = (v, g(v)) \in \mathrm{Graph}(g)$.

We conclude that the graph of g is closed, and so by the closed graph theorem that g is continuous. Similarly, the map f is continuous.

Question 4

(i) Define

$$S_n = 1 + x + x^2 + \cdots + x^n$$

Then

$$\|S_{n+k} - S_n\| = \|x^{n+1} + \cdots + x^{n+k}\| \leq \|x\|^{n+1}(1 + \|x\| + \cdots + \|x\|^{k-1})$$

so for all k

$$\|S_{n+k} - S_n\| \leq \|x\|^{n+1}(1 + \|x\| + \|x\|^2 + \cdots) = \frac{\|x\|^{n-1}}{1 - \|x\|}$$

since $\|x\| < 1$. If we let $n \to \infty$, we see $\|S_{n+k} - S_n\| \to 0$ for all k.

So the sequence (S_n) is a Cauchy sequence in A, and therefore converges to some element $y \in A$. But

$$S_n(1-x) = (1-x) + (x - x^2) + \cdots + (x^n - x^{n+1}) = 1 - x^{n+1}$$

Hence

$$\|S_n(1-x) - 1\| = \| - x^{n+1}\| \leq \|x\|^{n+1}$$

so

$$\|y(1-x) - 1\| = \lim_{n\to\infty} \|S_n(1-x) - 1\| = 0$$

since $\|x\| < 1$. Therefore $y(1-x) = 1$. Similarly $(1-x)y = 1$.

(ii) *Spectrum*(x) is the set of all $\lambda \in \mathbb{C}$ such that the element $x - \lambda 1 \in A$ is *not* invertible.

The spectral radius is defined by

$$R_\sigma(x) = \sup\{|\lambda| \mid \lambda \in \mathit{Spectrum}(x)\}$$

(iii) Let $\lambda \in \mathbb{C}$, $|\lambda| > \|x\|$. Set $y = \frac{1}{\lambda}x$. Then $\|y\| < 1$, so by part (i), $1 - y$ is invertible.

Hence, since $\lambda \neq 0$, $\lambda - x = \lambda(1-y)$ is invertible. So if $\lambda > \|x\|$, then $\lambda \notin \mathit{Spectrum}(x)$. We conclude that $R_\sigma(x) \leq \|x\|$.

(iv) We call U unitary if $UU^* = U^*U = I$, where I is the identity operator on H.

(v) Let $x \in H$. Then

$$\|Ux\|^2 = \langle Ux, Ux \rangle = \langle U^*Ux, x \rangle = \langle x, x \rangle = \|x\|^2$$

Hence $\|U\| = 1$.

(vi) By the above, $\|U\| = \|U^*\| = 1$. So if $|\lambda| > 1$, then $\lambda \notin \mathit{Spectrum}(U)$.

Let $\lambda \in \mathbb{C}$, $|\lambda| < 1$. Then by the above, the map $I - \lambda U^*$ is invertible in $\mathcal{L}(H)$. Multiplying by $-U$, we see that map $\lambda I - U$ is invertible, and again $\lambda \notin \mathit{Spectrum}(U)$.

Thus if $\lambda \in \mathit{Spectrum}(U)$, then $|\lambda| = 1$. The result now follows.

Question 5

(i) A linear map $K\colon V \to V$ is a *compact operator* if $\overline{K[B(0,1)_V]}$ is compact.
3 marks

(ii) Let $T\colon V \to W$ be a bounded linear map, with W finite-dimensional.

Since T is bounded, we have $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in V$. Thus, if $x \in B(0,1)_V$. then $\|Tx\| \leq M$.

Taking the closure, we see that if $y \in \overline{T[B(0,1)_V]}$ then $\|y\| \leq M$. So the set $\overline{T[B(0,1)_V]}$ is bounded. It is by definition closed. Since it is contained in a finite-dimensionsal space, it is compact by the Heine-Borel theorem. Thus T is a compact operator.

(iii) Let $T\colon V \to W$ be a linear map that is a compact operator. Then $\overline{T[B(0,1)_V]}$ is compact and therefore bounded. In particular, we have $M \geq 0$ such that $\|Tx\| \leq M$ whenever $\|x\| \leq \frac{1}{2}$.

So let $x \in V$. Then $x/2\|x\|$ has norm $\frac{1}{2}$, so $\|T(x/2\|x\|)\| \leq M$. Hence

$$\|Tx\| \leq 2M\|x\|$$

meaning T is a bounded linear map.

(iv) Let H be an infinite-dimensional Hilbert space. Let $I\colon H \to H$ be the identity map. Then $\overline{I[B(0,1)_H]} = \overline{B(0,1)_H} = \{v \in H \mid \|v\| \leq 1\}$. We claim this set is not compact.

To see this set is not compact, observe that we can choose an infinite orthonormal sequence $(e_n)_{n\in\mathbb{N}}$. This sequence lies within the set $\overline{B(0,1)_H}$, but has no convergent subsequence.

Hence the operator $I\colon H \to H$ is not compact. But as the identity map, it is certainly a bounded linear map.

(v) An operator $T \colon H \to H$ is *Fredholm* if $\ker T$ and $\ker T^*$ are finite-dimensional, and T has closed imagee. We define the *Fredholm index*

$$\mathrm{Ind}(T) = \dim\ker T - \dim\ker T^*$$

(vi) Prove that each of the following operators on the Hilbert space l^2 are Fredholm, and calculate their Fredholm indices. In the calculation, yiou may use any standard properties of the Fredholm index you need.

- Note that R is injective, so $\dim(\ker R) = 0$. We have

$$R^*(a_1, a_2, a_3, \dots) = (a_3, a_4, a_5, \dots)$$

so

$$\ker R^* = \{(a_1, a_2, 0, 0, 0, \dots) \mid a_i \in \mathbb{F}\}$$

and

$$\dim\ker R^* = 2$$

The image of R is certainly closed.
Thus R is Fredholm, and

$$\mathrm{Ind}(T) = \dim\ker T - \dim\ker T^* = -2$$

- Let $P \colon l^2 \to l^2$ be defined by

$$P(a_1, a_2, a_3, a_4, \dots) = (a_1, a_2, 0, 0, \dots)$$

Then P has finite-dimensional image, and is therefore a compact operator. It follows that $S = R + P$ is Fredholm, and

$$\mathrm{Ind}(S) = \mathrm{Ind}(R) = -2$$

- Let $Q \colon l^2 \to l^2$ be given by

$$Q(a_1, a_2, a_3, \dots) = (0, 0, -a_1, 0, 0, \dots)$$

Then as above, $T = R + Q$ is Fredholm, and

$$\mathrm{Ind}(T) = \mathrm{Ind}(R) = -2$$