Functional Analysis, Part 2

Paul D. Mitchener

Contents

1 Fourier and Wavelet Analysis **5**
1.1 The Space $L^2(\mathbb{R})$ 5
1.2 The Fourier Transform 9
1.3 Wavelets 13
1.4 The Wavelet Transform 15

2 Revision of Complex Analysis **19**
2.1 Path Integrals 19
2.2 Winding Numbers 21
2.3 Holomorphic Functions 22

3 Banach Algebras and Spectral Theory **23**
3.1 Banach Algebras 23
3.2 The Spectrum of an Element 25
3.3 The Resolvent 26

4 The Spectral Theory of Operators on Hilbert Space **29**
4.1 Unitary Operators 29
4.2 Self-Adjoint Operators 30
4.3 Compact Operators 32
4.4 Integral Kernels 35

5 Fredholm Operators **39**
5.1 The Notion of a Fredholm Operator 39
5.2 Properties of the Fredholm Index 42
5.3 The Toeplitz Index 46

Chapter 1

Fourier and Wavelet Analysis

1.1 The Space $L^2(\mathbb{R})$

Let $C \in \mathbb{R}$. Recall that we call a continuous function $f\colon [C, \infty) \to \mathbb{C}$ *integrable* if the limit

$$\int_C^\infty |f(t)|\ dt = \lim_{\alpha \to \infty} \int_C^a |f(x)|\ dx$$

exists and is finite.

Proposition 1.1 *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a continuous integrable function. Then the limit*

$$\int_C^\infty f(t)\ dt = \lim_{\alpha \to \infty} \int_C^\alpha f(x)\ dx$$

exists and is finite.

Proof: Let (α_n) be a monotonic increasing sequence such that $\alpha_n \to \infty$ as $n \to \infty$. Then

$$\lim_{n \to \infty} \int_C^{\alpha_n} |f(x)|\ dx = \int_C^\infty |f(x)|\ dx.$$

Let

$$\beta_n = \int_C^{\alpha_n} f(x)\ dx \qquad \gamma_n = \int_C^{\alpha_n} |f(x)|\ dx.$$

Let $n > m$. Then $\alpha_n \geq \alpha_m$, so

$$|\beta_n - \beta_m| = \left| \int_{\alpha_m}^{\alpha_n} f(x)\ dx \right| \leq \int_{\alpha_m}^{\alpha_n} |f(x)|\ dx = |\gamma_n - \gamma_m|.$$

Now, the sequence (γ_n) converges, and so is Cauchy. Hence the sequence (β_n) is also Cauchy. By completeness of the complex numbers, $\mathbb{C}$, it follows that it converges.

Now, let (α'_n) be another monotonic increasing sequence such that $\alpha'_n \to \infty$ as $n \to \infty$. Let

$$\beta'_n = \int_C^{\alpha'_n} f(x)\ dx \qquad \gamma'_n = \int_C^{\alpha'_n} |f(x)|\ dx.$$

Then

$$\lim_{n\to\infty} [\gamma'_n] = \lim_{n\to\infty} \int_C^{\alpha_n} |f(x)|\ dx = \lim_{n\to\infty} [\gamma_n]$$

so $|\gamma_n - \gamma'_n| \to 0$ as $n \to \infty$. As above, $|\beta'_n - \beta_n| \leq |\gamma_n - \gamma'_n|$. Therefore

$$\lim_{n\to\infty} \int_C^{\alpha_n} f(x)\ dx = \lim_{n\to\infty} [\beta'_n] = \lim_{n\to\infty} [\beta'_n] = \lim_{n\to\infty} \int_C^{\alpha'_n} f(x)\ dx.$$

It follows that the limit

$$\int_C^{\infty} f(t)\ dt = \lim_{\alpha\to\infty} \int_C^{\alpha} f(x)\ dx$$

is well-defined. □

Example 1.2 *Observe* $|e^{-x}| = e^{-x}$. *We have*

$$\int_0^{\infty} e^{-x}\ dx = \lim_{C\to\infty} \left[-e^{-x}\right]_1^C = \lim_{C\to\infty} (1 - e^{-C}) = 1.$$

In particular, e^{-x} *is integrable.*

We similarly term a continuous function $f\colon (-\infty, C]$ *integrable* if the limit

$$\int_{-\infty}^{C} |f(t)|\ dt = \lim_{\alpha\to-\infty} \int_{\alpha}^{C} |f(t)|\ dt$$

exists and is finite. In this case, we define

$$\int_{-\infty}^{C} f(t)\ dt = \lim_{\alpha\to-\infty} \int_{\alpha}^{C} f(t)\ dt$$

A continuous function $f\colon \mathbb{R} \to \mathbb{C}$ is termed *integrable* if the restrictions $f|_{[0,\infty)}$ and $f|_{(-\infty,0]}$ are both integrable. The integral is defined by writing

$$\int_{-\infty}^{\infty} f(t)\ dt = \int_{-\infty}^{0} f(t)\ dt + \int_0^{\infty} f(t)\ dt = \lim_{\alpha,\beta\to\infty} \int_{-\alpha}^{\beta} f(t)\ dt.$$

Example 1.3 *We say a function* $f\colon \mathbb{R} \to \mathbb{C}$ *has* compact support *if we have an interval* $[a, b] \subseteq \mathbb{R}$ *such that* $f(x) = 0$ *if* $x \notin [a, b]$.

In this case, the function f *is integrable, and*

$$\int_{-\infty}^{\infty} f(t)\ dt = \int_a^b f(t)\ dt.$$

The following result is left as an exercise.

Proposition 1.4 *Let $L^1_{\text{cont}}(\mathbb{R})$ be the space of continuous integrable functions $f\colon \mathbb{R} \to \mathbb{C}$. Then we have a norm on $L^1_{\text{cont}}(\mathbb{R})$ defined by the formula*

$$\|f\| = \int_{-\infty}^{\infty} |f(t)|\ dt.$$

□

Definition 1.5 *We write $L^1(\mathbb{R})$ to denote the completion of the space $L^1_{\text{cont}}(\mathbb{R})$ witth respect to the above norm.*

Definition 1.6 *We call a continuous function $f\colon \mathbb{R} \to \mathbb{C}$* L^2-integrable *if the function $|f|^2$ is integrable, that is*

$$\int_{-\infty}^{\infty} |f(t)|^2\ dt < \infty.$$

Let $L^2_{\text{cont}}(\mathbb{R})$ denote the space of L^2-integrable continuous functions. The following is left as an exercise.

Proposition 1.7 *We have an inner product on the space $L^2_0(\mathbb{R})$ defined by the formula*

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \overline{f(t)} g(t)\ dt.$$

□

Definition 1.8 *We write $L^2(\mathbb{R})$ to denote the Hilbert space obtained by completion of the space $L^2_{\text{cont}}(\mathbb{R})$ with respect to the norm defined by the above inner product.*

Now, let $f\colon (a,b) \to \mathbb{C}$ be a continuous function where the one-sided limits $\lim_{t\to a^+} f(t)$ and $\lim_{t\to b^-} f(t)$ exist. Then it is straightforward to check that we can define the integral

$$\int_a^b f(t)\ dt = \lim_{\substack{d\to b^+ \\ c\to a^-}} \int_c^d f(t)\ dt.$$

We say a function $f\colon \mathbb{R} \to \mathbb{C}$ has a *jump discontinuity* at $c \in \mathbb{R}$ if the one-sided limits

$$\lim_{t\to c^+} f(t) \qquad \lim_{t\to c^-} f(t)$$

both exist, but are not equal.

Definition 1.9 *We call a function $f\colon \mathbb{R} \to \mathbb{C}$* piecewise-continuous *if it is continuous apart from finitely many jump discontinuities.*

As well as continuous functions, we can integrate piecewise-continuous functions. To be precise, if $f\colon \mathbb{R} \to \mathbb{C}$ is a function with finitely many jump discontinuities, $a_1 < a_2 < \cdots < a_k$, then we define

$$\int_{-\infty}^{\infty} f(t)\ dt = \int_{-\infty}^{a_1} f(t)\ dt + \int_{a_1}^{a_2} f(t)\ dt + \cdots + \int_{a_{k-1}}^{a_k} f(t)\ dt + \int_{a_k}^{\infty} f(t)\ dt.$$

assuming the restrictions $f_{(-\infty,a_1]}$ and $f_{[a_k,\infty)}$ are integrable.

Example 1.10 *Let I be one of the intervals (a,b), $(a,b]$, $[a,b)$ or $[a,b]$. Define the* characteristic function *of I, $\chi_I\colon \mathbb{R} \to \mathbb{C}$, by writing*

$$\chi_I(t) = \begin{cases} 1 & t \in I \\ 0 & t \notin I \end{cases}$$

Then

$$\int_{-\infty}^{\infty} f(t)\ dt = b - a.$$

We call a piecewise-continuous function $f\colon \mathbb{R} \to \mathbb{C}$ L^2*-integrable* if, as before

$$\int_{-\infty}^{\infty} |f(t)|^2\ dt < \infty.$$

It is a fact that any piecewise-continuous L^2-integrable function is a norm-limit of continuous L^2-integrable functions. Thus we can also regard such functions as belonging to the space $L^2(\mathbb{R})$. Similarly, integrable piecewise-continuous functions can be regarded as belonging to $L^1(\mathbb{R})$. In fact, we need to consider such functions, rather than just continuous functions, for some examples in this chapter.

Definition 1.11 *Let $I_1, \ldots, I_n$ be intervals, where I_j is one of the intervals (a_j,b_j), $(a_j,b_j]$, $[a_j,b_j)$ or $[a_j,b_j]$.*

Let $c_k \in \mathbb{C}$. Then we call the function

$$s = c_1\chi_{I_1} + \cdots + c_n\chi_{I_n}$$

a step function.

Note that, by linearity of the integral, we have

$$\int_{-\infty}^{\infty} s(t)\ dt = c_1(b_1 - a_1) + \cdots + c_n(b_n - a_n).$$

Now, step functions and their integrals are fundamental to the more sophisticated set-up of integration theory known as *Lebesgue integration*. We do not consider Lebesgue integration here, but we need to note the following result.

Proposition 1.12 *The set of step functions is a dense subset of the space $L^2(\mathbb{R})$.* □

1.2 The Fourier Transform

Definition 1.13 *Let $f \in L^1(\mathbb{R})$ be piecewise-continuous. Then we define the* Fourier transform *of f to be the function $f\colon \mathbb{R} \to \mathbb{C}$ defined by the formula*

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\ dx.$$

The alternative notation $\mathfrak{F}\{f(x)\}$ is sometimes used instead of the symbol $\hat{f}$ to denote a Fourier transform. This alternative notation is particularly useful if we have a formula describing a function.

Example 1.14

$$\mathfrak{F}\{e^{-x^2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\left(x + i\frac{\omega}{2} \right)^2 - \frac{\omega^2}{4} \right)\ dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4} \int_{-\infty}^{\infty} \exp\left(-\left(x + i\frac{\omega}{2} \right)^2 \right)\ dx = \frac{1}{\sqrt{2}} e^{-\omega^2/4}.$$

The following properties of the Fourier transform are straightforward to check; we leave them as exercises.

Proposition 1.15 *Let $f \in L^1(\mathbb{R})$ be piecewise-continuous. Let $\alpha \in \mathbb{R}$. Then*

- $\mathfrak{F}\{e^{i\alpha x} f(x)\} = \hat{f}(\omega - \alpha)$.
- $\mathfrak{F}\{f(x - x_0)\} = \hat{f}(\omega) e^{-i\omega x_0}$.
- *Let $\alpha > 0$. Then* $\mathfrak{F}\{f(\alpha x)\} = \frac{1}{\alpha} \hat{f}\left(\frac{\omega}{\alpha}\right)$.
- $\mathfrak{F}\{\overline{f(x)}\} = \overline{\mathfrak{F}\{f(-x)\}}$.
- *Let $\alpha, \beta \in \mathbb{C}$ and g be another piecewise-continuous function in $L^1(\mathbb{R})$. Then* $\mathfrak{F}\{\alpha f(x) + \beta g(x)\} = \alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$.

□

The following is also straightforward to check using integration by parts.

Proposition 1.16 *Let $f \in L^1(\mathbb{R})$ be piecewise-differentiable. Suppose $f' \in L^1(\mathbb{R})$, and $\lim_{|x|\to\infty} f(x) = 0$. Then*

$$\mathfrak{F}\{f'(x)\} = i\omega \hat{f}(\omega).$$

□

Proposition 1.17 *Let $f, g \in L^1(\mathbb{R})$ be piecewise-continuous. Let $x \in \mathbb{R}$. Then the integral*

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - t) g(t)\ dt$$

exists, and is finite. Further, the function $f * g$ is integrable.

Proof: We need to show that the double integral

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-t)g(t)|\ dt\ dx$$

exists and is finite. Since the functions f and g have only jump discontinuities, we can change the order of integration to see

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x-t)g(t)|\ dx\ dt = \int_{-\infty}^{\infty} |g(t)| \int_{-\infty}^{\infty} |f(x-t)|\ dx\ dt.$$

Making the substitution $y - x - t$, we see

$$I = \left(\int_{-\infty}^{\infty} |g(t)|\ dt \right) \left(\int_{-\infty}^{\infty} |f(y)|\ dy \right)$$

which is a well-defined finite quantity.

In particular, $(f*g)(x)$ exists, and is finite, and forms an integrable function. □

We call $f * g$ the *convolution* of the functions f and g. The following result is known as the *convolution formula.*

Proposition 1.18 *Let $f, g \in L^1(\mathbb{R})$ be piecewise-continuous. Then*

$$\mathfrak{F}\{f * g(x)\} = \mathfrak{F}\{f(x)\}\mathfrak{F}\{g(x)\}.$$

Proof: By definition

$$\mathfrak{F}\{f * g(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \int_{-\infty}^{\infty} f(x-t)g(t)\ dt\ dx.$$

Exchanging the order of integration

$$\mathfrak{F}\{f * g(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) \int_{-\infty}^{\infty} f(x-t)e^{-i\omega t}\ dt\ dx.$$

Making the substitution $y = x - t$, we see

$$\mathfrak{F}\{f * g(x)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) \int_{-\infty}^{\infty} f(y)e^{-i\omega y}e^{-i\omega t}\ dt\ dx.$$

But the above integral is equal to $\hat{f}(\omega)\hat{g}(\omega)$ as desired. □

Actually, we would like to define the Fourier transform on $L^2(\mathbb{R})$ rather than on $L^1(\mathbb{R})$. The problem is that integrable functions are not necessarily L^2-integrable, nor L^2-integrable functions necessarily integrable.

Definition 1.19 *We call a function $f: \mathbb{R} \to \mathbb{C}$* compactly supported *if we have an interval $[a, b]$ such that $f(t) = 0$ if $t \notin [a, b]$.*

We call the smallest interval $[a, b]$ with the above property the *support* of f.

Certainly, any compactly-supported piecewise-continuous function is integrable, with integral

$$\int_{-\infty}^{\infty} f(t)\ dt = \int_a^b f(t)\ dt.$$

where $[a, b]$ is the support of f. If f is compactly supported, then so is $|f|^2$, so any compactly supported piecewise-continuous function is L^2-integrable.

We remarked at the end of the previous section that the step functions are a dense subset of $L^2(\mathbb{R})$. Certainly, any step function is compactly supported. Therefore the compactly supported piecewise-continuous functions form a dense subset, $L^2_c(\mathbb{R})$, of $L^2(\mathbb{R})$.

If $f \in L^2_c(\mathbb{R})$, then f is certainly integrable, so the the Fourier transform, $\hat{f}$, is well-defined.

Lemma 1.20 *Let $f \in L^2_c(\mathbb{R})$. Then, with respect to the L^2-norm, $\|\hat{f}\| = \|f\|$.*

Proof: Without loss of of generality, suppose that f has support in the interval $[0, 2\pi]$. Then $f \in L^2[0, 2\pi]$. Let $\xi \in \mathbb{R}$, and set $g(t) = e^{-i\xi t}$.

Recall that the space $L^2[0, 2\pi]$ has an orthonormal basis

$$\{\phi_k \mid k \in \mathbb{Z}\} \qquad \phi_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}.$$

Hence, by Parseval's theorem

$$\|g\|^2 = \sum_{k=-\infty}^{\infty} |\langle \phi_k, g\rangle|^2.$$

But

$$\langle \phi_k, g\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-ikt} g(t)\ dt = \hat{g}(k)$$

so

$$\|g\|^2 = \sum_{k=-\infty}^{\infty} |\hat{g}(k)|^2.$$

But $\|g\| = \|f\|$, and $\hat{g}(k) = \hat{f}(k + \xi)$, so

$$\|f\|^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k + \xi)|^2.$$

Now integrate both sides between 0 and 1. We get

$$\|f\|^2 = \sum_{k=-\infty}^{\infty} \int_0^1 |\hat{f}(k + \xi)|^2\ d\xi = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\ d\xi = \|\hat{f}\|^2$$

and we are done. □

Thus the Fourier transform is an isometry $\mathfrak{F}\colon L_c^2(\mathbb{R}) \to L^2(\mathbb{R})$. As $L_c^2(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R})$, the Fourier transform extends uniquely to an isometry $\mathfrak{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$, which we again call the Fourier transform.

Actually, it turns out that the Fourier transform is a unitary operator. Before we prove this, we need a formula for the inverse of a Fourier transform; the establish this inversion formula, we need two technical lemmas.

Lemma 1.21 *Let $f, g \in L_c^2(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\ dx = \int_{-\infty}^{\infty} \hat{f}(x)g(x)\ dx.$$

□

The proof is an exercise.

Lemma 1.22 *Let $f \in L_c^2(\mathbb{R})$ and $g = \overline{\mathfrak{F}\{f\}}$. Then $f = \overline{\mathfrak{F}\{g\}}$.*

Proof: By the above lemma and lemma 1.20, we have

$$\langle f, \overline{\mathfrak{F}\{g\}} \rangle = \langle \mathfrak{F}\{f\}, \overline{g} \rangle = \langle \hat{f}, \hat{f} \rangle = \|\hat{f}\|^2 = \|f\|^2.$$

Taking complex conjugates,

$$\langle \overline{\mathfrak{F}\{g\}}, f \rangle = \|f\|^2.$$

By lemma 1.20, we have

$$\|\hat{g}\|^2 = \|g\|^2 = \|\overline{\mathfrak{F}\{f\}}\|^2 = \|f\|^2.$$

Using the above three equations

$$\|f - \overline{\mathfrak{F}\{g\}}\|^2 = \langle f - \overline{\mathfrak{F}\{g\}}, f - \overline{\mathfrak{F}\{g\}} \rangle = \|f\|^2 - \|f\|^2 - \|f\|^2 + \|f\|^2 = 0.$$

The result now follows. □

The following important result is called the *Fourier inversion formula.*

Theorem 1.23 *Let $f \in L_c^2(\mathbb{R})$. Then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{f}(\omega)\ d\omega.$$

Proof: Let $g = \overline{\mathfrak{F}\{f\}}$. Then by the above lemma

$$f(x) = \overline{\mathfrak{F}\{g\}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \overline{g(\omega)}\ d\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{f}(\omega)\ d\omega.$$

□

Example 1.24 *Let*

$$\hat{\phi}(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}} & -\pi \leq \omega < \pi \\ 0 & \textit{otherwise.} \end{cases}$$

Then

$$\phi(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega x}\, d\omega = \frac{\sin(\pi x)}{\pi x}.$$

Corollary 1.25 *The Fourier transform $\mathfrak{F}\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary operator.*

Proof: Let $f \in L^2_c(\mathbb{R})$. Then

$$\mathfrak{F}^*(f)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega x} f(\omega)\, dx$$

so by the above

$$\mathfrak{F}^*(\hat{f}) = f.$$

Extending to $L^2(\mathbb{R})$, we see that $\mathfrak{F}^*\mathfrak{F} = I$. Similarly, $\mathfrak{F}\mathfrak{F}^* = I$. □

1.3 Wavelets

Recall that if we have a Hilbert space H, and a basis (e_n), then an element $v \in H$ can be written as a sum

$$v = \sum_{n=1}^{\infty} \alpha_n e_n \qquad \alpha_n = \langle e_n, v\rangle.$$

The classic example comes from Fourier series. In the Hilbert space $L^2[0, 2\pi]$, every element $f \in L^2[0, 2\pi]$ can be written as a sum

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)).$$

All of these functions can be viewed as dilations and translations of the basic function $\sin x$. Thus a function on $L^2[0, 2\pi]$ can be viewed as a sum of waves of various frequencies.

On the other hand, we cannot obtain a basis for the space $L^2(\mathbb{R})$ by looking at translations and dilations of $\sin x$ as the function $\sin x$ is itself not integrable.

Our idea in this section is to build orthonormal bases of the space $L^2(\mathbb{R})$ by looking at translations and dilations of a 'wave packet'- a finite or at least integrable oscillation.

Definition 1.26 *Let* $\psi \in L^2(\mathbb{R})$ *and* $j, k \in \mathbb{Z}$. *Then we define the function* $\psi_{jk} \in L^2(\mathbb{R})$ *by the formula*

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k).$$

We call ψ *an* orthonormal wavelet *if the set* $\{\psi_{jk} \mid j, k \in \mathbb{Z}\}$ *is an orthonormal basis for* $L^2(\mathbb{R})$.

Note that if $f \in L^2(\mathbb{R})$ and ψ is an orthonormal wavelet, then

$$f = \sum_{j,k=-\infty}^{\infty} \alpha_{jk}\psi_{jk} \qquad \alpha_{jk} = \langle \psi_{jk}, f \rangle.$$

This series is called the *wavelet series* of f. The function ψ is sometimes called the *mother wavelet* of the series.

Proposition 1.27 *Let*

$$H(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} \le x < 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Then H *is an orthonormal wavelet.*

Proof: Let $j, k \in \mathbb{Z}$. Then

$$H_{jk}(x) = \begin{cases} 2^{j/2} & 2^{-j}k < x \le 2^{-j}(k + \frac{1}{2}) \\ -2^{j/2} & 2^{-j}(k + \frac{1}{2}) < x \le 2^{-j}(k+1) \\ 0 & \text{otherwise.} \end{cases}$$

We need to show that $\{H_{jk} \mid j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. Observe:

- $$\int_{-\infty}^{\infty} H_{jk}(x)\ dx = 0 \qquad \int_{-\infty}^{\infty} |H_{jk}(x)|^2\ dx = 1.$$

- For fixed j, the supports of H_{jk} and $H_{jk'}$ are disjoint if $k \neq k'$, so the set $\{H_{jk} \mid k \in \mathbb{Z}\}$is orthonormal.

- Let $i > j$. Then the function H_{ik} is constant on the support of $H_{jk'}$ for all k and k'. Hence $\langle H_{ik}, H_{jk'} \rangle = 0$.

It follows that the set $\{H_{jk} \mid j, k \in \mathbb{Z}\}$ is orthonormal.

Let $m, n \in \mathbb{Z}$. Let $X_{m,n}$ be the characteristic function of the interval $(n2^m, (n+1)2^m]$. Note that for fixed m, these intervals are disjoint for different values of n.

If $m \geq j$, then there is a unique integer n such that the support of $H_{j,k}$ is contained in the interval $(n2^m, (n+1)2^m]$. Thus $\langle X_{m,n}, H_{j,k} \rangle = 0$.

Let $m < j$. Again looking at overlaps of intervals, there is a unique integer n such that either the positive half or the negative half of the function H_{jk} is contained in the interval $(n2^m, (n+1)2^m]$. For all other values n, the function H_{jk} does not overlap with the interval, and so $\langle X_{m,n}, H_{j,k} \rangle = 0$.

For the particular special value of n, we have $\langle X_{m,n}, H_{j,k} = \pm 2^m 2^{-j/2}$.

Combining all this, we see that

$$\sum_{j,k=-\infty}^{\infty} |\langle X_{m,n}, H_{j,k} \rangle|^2 = 2^{2m} \sum_{j=m+1}^{\infty} 2^{-j} = 2^{2m} = \|X_{m,n}\|^2.$$

Hence, by Parseval's theorem, the function $X_{m,n}$ belongs to the span of the functions H_{jk}.

Now, let S be the set of step functions with discontinuities at the rational numbers of the form $n2^m$ with $m, n \in \mathbb{Z}$. Then S is a dense subspace of the space of all step functions. Further, S is the span of the set of all the functions $X_{m,n}$.

Thus, by the above, the span of the functions H_{jk} contains S, which is a dense subset of the space of all step functions, and hence a dense subset of $L^2(\mathbb{R})$. Thus the span of the orthonormal set $\{H_{jk} \mid j, k \in \mathbb{Z}\}$ is a dense subset of $L^2(\mathbb{R})$, and so it is an orthonormal basis for $L^2(\mathbb{R})$. □

The orthonormal wavelet H is called the *Haar wavelet*.

1.4 The Wavelet Transform

The theory of orthonormal wavelets is inspired by the theory of Fourier series. The work in this section is inspired by the Fourier transform.

Definition 1.28 *An* admissible wavelet *is a piecewise-continuous function* $\psi \in L^2(\mathbb{R})$ *such that*

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, d\omega < \infty$$

where $\hat{\psi}$ *is the Fourier transform.*

This technical condition is needed so that an inversion formula, given below, makes sense. All will become clear!

Example 1.29 *Consider the Haar wavelet from the previous section*

$$H(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x < 1 \\ 0 & \textit{otherwise.} \end{cases}$$

We can show that

$$\hat{H}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\sin^2(\omega/4)}{\omega/4} e^{-(\omega-\pi)/2}$$

and

$$\int_{-\infty}^{\infty} \frac{\hat{H}(\omega)}{|\omega|} d\omega = \frac{8}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(\omega/4)}{|\omega|^3}\ d\omega < \infty.$$

So H is an admissible wavelet.

The following technical result gives us a way to construct new admissible wavelets. We omit the proof.

Proposition 1.30 *Let ψ be an admissible wavelet. Let $\phi \in L^1(\mathbb{R})$ be a bounded piecewise-continuous function. Then the convolution $\psi * \phi$ is an admissible wavelet.* □

We now come to the main definition of this section.

Definition 1.31 *Let $\psi \in L^2(\mathbb{R})$ be an admissible wavelet. Let $f \in L^2(\mathbb{R})$. Then we define the* wavelet transform *by the formula*

$$W_\psi f(s,t) = \frac{1}{\sqrt{|s|}} \int_{-\infty}^{\infty} \overline{\psi\left(\frac{x-t}{s}\right)} f(x)\ dx$$

where $s, t \in \mathbb{R}$ and $s \neq 0$.

The function ψ is sometimes called the *mother wavelet* of the transformation.

Let $\alpha_{j,k} = W_\psi f(2^{-j}, k2^{-j})$. Then by definition, $\alpha_{j,k} = \langle \psi_{j,k}, f \rangle$. In particular, if ψ is an orthonormal wavelet, then

$$f = \sum_{j,k=-\infty}^{\infty} \alpha_{j,k} \psi_{j,k}.$$

The following result litst some elementary properties of the wavelet transform; the proof is left as a straightforward exercise.

Proposition 1.32 *Let ψ and ϕ be wavelets, and let $f, g \in L^2(\mathbb{R})$. Then:*

- $W_\psi(\alpha f + \beta g)(s,t) = \alpha W_\psi f(s,t) + \beta W_\psi g(s,t)$ *for all $\alpha, \beta \in \mathbb{C}$.*
- *Let $c \in \mathbb{R}$. Let T_c: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the* translation operator *defined by the formula $T_c f(t) = f(t-c)$. Then*

$$W_\psi(T_c f)(s,t) = (W_\psi f)(s, t-c).$$

- *Let $c > 0$. Let D_c: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the* dilation operator *defined by the formula $D_c f(t) = \frac{1}{c} d(\frac{t}{c})$. Then*

$$W_\psi(D_c f)(s,t) = \frac{1}{\sqrt{c}} (W_\psi f)(\frac{s}{c}, \frac{t}{c}).$$

- $W\psi\phi(s,t) = \overline{W_\phi\psi(\frac{1}{s}, -\frac{t}{s})}$.
- $W_{\alpha\phi+\beta\psi}f(s,t) = \overline{\alpha}W_\psi f(s,t) + \overline{\beta}W_\phi f(s,t)$ *for all* $\alpha, \beta \in \mathbb{C}$.
- *Let* $P\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *be the* parity operator *given by writing* $Pf(t) = f(-t)$. *Then*
$$W_{P\psi}Pf(s,t) = W_\psi f(s,-t).$$
- *Let* $c \in R$. *Then* $W_{T_c\psi}f(s,t) = W_\psi(s, t+cs)$.
- *Let* $c > 0$. *Then* $W_{D_c\psi}f(s,t) = \frac{1}{\sqrt{c}}(W_\psi f)(cs,t)$.

□

We now present a technical lemma needed for the inversion formula for the wavelet transform. In this lemma, the admissibility condition is very much present.

Lemma 1.33 *Let* ψ *be an admissible wavelet. Set*

$$C_\psi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\ d\omega.$$

Then for piecewise-continuous functions $f, g \in L^2(\mathbb{R})$, *we have*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\psi f(s,t)\overline{W_\psi g(s,t)}\ \frac{dt\ ds}{s^2} = C_\psi\langle f, g\rangle.$$

Proof: Set
$$\psi_{s,t}(x) = \frac{1}{\sqrt{|s|}}\psi\left(\frac{x-t}{s}\right)$$
so that $W_\psi f(s,t) = \langle\psi_{s,t}, f\rangle$. Since the Fourier transform is a unitary operator on $L^2(\mathbb{R})$, we have
$$W_\psi f(s,t) = \langle\hat{\psi_{s,t}}, \hat{f}\rangle.$$

By the elementary properties of the Fourier transform, we have

$$\hat{\psi_{s,t}}(\omega) = \sqrt{|s|}e^{-it\omega}\hat{\psi}(s\omega);$$

the details are left as an exercise. Thus

$$W_\psi f(s,t) = \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{\psi}(sx)}\sqrt{|s|}e^{itx}\ dx$$

that is to say
$$W_\psi f(s,t) = \sqrt{2\pi s}\mathfrak{F}\{\hat{f}(x)\overline{\hat{\psi}(sx)}(-t).$$

Similarly
$$\overline{W_\psi g(s,t)} = \sqrt{2\pi s}\overline{\mathfrak{F}\{\hat{g}(x)\overline{\hat{\psi}(sx)}(-t)}.$$

Now, using the above and the Fourier inversion formula

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\psi f(s,t)\overline{W_\psi g(s,t)}\,\frac{dt\;ds}{s^2} = 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathfrak{F}\{\hat{f}(x)\overline{\hat{\psi}(sx)}(-t)\overline{\mathfrak{F}\{\hat{g}(x)\overline{\hat{\psi}(sx)}(-t)}\,dt\frac{ds}{s}$$

$$= 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}(x)}|\hat{\psi}(sx)|^2 dx\frac{ds}{s}$$

$$= 2\pi \int_{-\infty}^{\infty} |\hat{\psi}(sx)|^2\frac{ds}{s}\int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}(x)}dx = C_\psi\langle f,g\rangle$$

and we are done. Our integrability conditions are sufficient to allow us to swap the double integrals needed in the above calculation. □

Our next, and final result is known as the *inversion formula* for the wavelet transform. Note again that the admissibility condition is needed for the result to make sense.

Theorem 1.34 *Let $f \in L^2(\mathbb{R})$ be piecewise-continuous. Then*

$$f(x) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\psi f(s,t)\psi_{s,t}(x)\;\frac{dt\;ds}{s^2}$$

where

$$\psi_{s,t}(x) = \frac{1}{\sqrt{|s|}}\psi\left(\frac{x-t}{s}\right).$$

Proof: By the above lemma

$$C_\psi\langle f,g\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(s,t)\overline{(W_\psi g)(s,t)}\frac{dt\;ds}{s^2}.$$

Now

$$\overline{W_\psi g(s,t)} = \int_{-\infty}^{\infty} \overline{g(u)}\psi_{s,t}(u)\;du$$

so we can rewrite the above as

$$C_\psi\langle f,g\rangle = \langle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (W_\psi f)(s,t)\psi_{s,t}\frac{dt\;ds}{s^2},g\rangle.$$

Since this formula holds for all g, it follows that

$$f(x) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\psi f(s,t)\psi_{s,t}(x)\;\frac{dt\;ds}{s^2}$$

as required. □

Chapter 2

Revision of Complex Analysis

Most of what is in this chapter is revision from earlier work in complex analysis.

2.1 Path Integrals

Let $D \subseteq \mathbb{C}$ be an open subset. A *path* in D is a continuous piecewise-differentiable map $\gamma\colon [a,b] \to D$. Here, *piecewise-differentiable* means we can form the derivative $\gamma'(t) \in \mathbb{C}$ for all but finitely many $t \in [a,b]$. The following definition therefore makes sense.

Definition 2.1 *Let $f\colon D \to \mathbb{C}$ be a continuous function. Then we define the* path integral

$$\int_\gamma f(z)\ dz = \int_a^b \gamma'(t) f(\gamma(t))\ dt.$$

If γ is a *closed contour*, that is to say $\gamma(a) = \gamma(b)$, we sometimes write the above integral using the notation

$$\oint_\gamma f(z)\ dz$$

We now present a number of results to help us evaluate and analyse path integrals.

Proposition 2.2 *Let $\alpha, \beta \in \mathbb{C}$, and let $f, g\colon D \to \mathbb{C}$ be continuous. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise-differentiable path. Then*

$$\int_\gamma \alpha f(z) + \beta g(z)\ dz = \alpha \int_\gamma f(z)\ dz + \beta \int_\gamma g(z)\ dz$$

Proof: The result follows immediately be definition of the path integral and linearity of the integral of a complex-valued function. □

Definition 2.3 *Let* $\gamma\colon [a,b] \to D$ *and* $\beta\colon [b,c] \to D$ *be piecewise-differentiable paths such that* $\gamma(b) = \beta(b)$*. Then we define the* concatenation *of* γ *and* β *to be the path* $\gamma\beta\colon [a,c] \to D$ *defined by writing*

$$\gamma\beta(t) = \begin{cases} \gamma(t) & t \leq b \\ \beta(t) & t \geq b \end{cases}$$

Note that by definition the concatenation of two piecewise-differentiable paths is a piecewise-differentiable path.

Note that, even when the paths $\gamma\colon [a,b] \to D$ and $\beta\colon [b,c] \to D$ are differentiable, the concatenation $\gamma\beta$ is probably not differentiable, but only piecewise-differentiable; in general, we have a 'kink' at the point b.

The following result is completely straightforward to check.

Proposition 2.4 *Let* $f\colon D \to \mathbb{C}$ *be continuous, and let* $\gamma\colon [a,b] \to D$ *and* $\beta\colon [b,c] \to D$ *be piecewise-differentiable paths such that* $\gamma(b) = \beta(b)$*. Then*

$$\int_{\gamma\beta} f(z)\ dz = \int_{\gamma} f(z)\ dz + \int_{\beta} f(z)\ dz$$

□

Definition 2.5 *Let* $\gamma\colon [a,b] \to D$ *be a continuous path. Then we define the* reverse path*,* $\overline{\gamma}\colon [a,b] \to D$*, by the formula*

$$\overline{\gamma}(t) = \gamma(a+b-t)$$

The reverse of a piecewise-differentiable path is again piecewise-differentiable. The following result is again straightforward.

Proposition 2.6 *Let* $\gamma\colon [a,b] \to D$ *be a piecewise-differentiable path, and let* $f\colon D \to \mathbb{C}$ *be continuous. Then*

$$\int_{\overline{\gamma}} f(z)\ dz = -\int_{\gamma} f(z)\ dz$$

□

The following result shows that we can reparametrise paths when evaluating integrals. We leave the proof as an exercise.

Proposition 2.7 *Let* $\gamma\colon [a,b] \to D$ *be a piecewise-differentiable path, and let* $\theta\colon [c,d] \to [a,b]$ *be a differentiable mapping such that* $\theta(c) = a$ *and* $\theta(d) = b$*.*

Let $f\colon D \to \mathbb{C}$ *be continuous. Then*

$$\int_{\gamma} f(z)\ dz = \int_{\gamma\circ\theta} f(z)\ dz$$

□

Proposition 2.8 *Let $\gamma\colon [a,b] \to D$ be a piecewise-differentiable path. Let $f\colon D \to \mathbb{C}$ be continuous. Then*

$$\left|\int_\gamma \omega\right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\ dt$$

Proof: By the mean value theorem for integrals, we have

$$\begin{array}{rcl} |\int_\gamma f(z)\ dz| & = & |\int_a^b f(\gamma(t), \gamma'(t))\ dt| \\ & \leq & \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\ dt \end{array}$$

□

2.2 Winding Numbers

Definition 2.9 *Let $w \in \mathbb{C}$. Let $\gamma\colon [0,1] \to \mathbb{C}\backslash\{w\}$ be a closed contour.*
Then we define winding number *of γ about w to be:*

$$Wind(\gamma; w) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z-w} dz$$

Let $D \subseteq \mathbb{C}$ be open. We call two paths $\gamma_1, \gamma_2\colon [a,b] \to D$ with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$ *homotopic* if there is a continuous map $H\colon [a,b] \times [0,1] \to D$ such that:

- $H(t,0) = \gamma_1(t)$ for all $t \in [a,b]$.
- $H(t,1) = \gamma_2(t)$ for all $t \in [a,b]$.
- $H(0,s) = \gamma_1(0) = \gamma_2(0)$ for all $s \in [0,1]$.
- $H(1,s) = \gamma_1(1) = \gamma_2(1)$ for all $s \in [0,1]$.

The fundamental properties of the winding number are contained in the following theorem.

Theorem 2.10 *Let $w \in \mathbb{C}$.*

- *Let γ be a circle which winds once anticlockwise around w. Then $Wind(\gamma; w) = 1$.*
- *Let $gamma_1\colon [a,b] \to \mathbb{C}\backslash\{w\}$ and $\gamma_2\colon [b,c] \to \mathbb{C}\backslash\{w\}$ be closed contours such that $\gamma_1(b) = \gamma_2(b)$. Then $Wind(\gamma_2\gamma_1; w) = Wind(\gamma_1; w) + Wind(\gamma_2; w)$.*
- *Let $\gamma_1, \gamma_2\colon [a,b] \to \mathbb{C}\backslash\{w\}$ be homotopic closed contours. Then $Wind(\gamma_1; w) = Wind(\gamma_2; w)$.*

□

Only the last of these properties is significantly difficult to prove. Anyway, the idea that $Wind(\gamma; w) = 1$ means visually that the path γ winds once anticlockwise around the point $w \in \mathbb{C}$.

2.3 Holomorphic Functions

We can talk about holomorphic functions with values in complex Banach spaces as well as with values in the complex numbers. Specifically, we have the following definition.

Definition 2.11 *Let $D \subseteq \mathbb{C}$ be a connected open set. Let V be a complex Banach space. We call a function $f: D \to V$* holomorphic *if there is a function $f': D \to V$ such that*

$$\lim_{h \to 0} \left\| \frac{f(z+h) - f(z)}{h} - f'(z) \right\| = 0$$

The function f' is called the derivative *of f.*

Just as in the classical case, the following holds.

Proposition 2.12 *Let $f: D \to V$ be holomorphic. Then f is continuous.* □

Recall the following results from complex analysis.

Theorem 2.13 (Cauchy's Theorem) *Let $f: D \to \mathbb{C}$ be holomorphic. Let γ be a closed contour in D such that $Wind(\gamma; w) = 0$ whenever $w \in \mathbb{C} \backslash D$. Then*

$$\oint_\gamma f(z)\ dz = 0.$$

□

Theorem 2.14 (Cauchy's Integral Formula) *Let $f: D \to \mathbb{C}$ be holomorphic. Let $w \in D$, and let γ be a closed contour in D such that $Wind(\gamma; w) = 1$.*
Then:

$$f(w) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-w)}\ dz$$

□

The following result is known as *Liouville's Theorem*, and can be deduced (non-trivially) from Cauchy's integral formula.

Theorem 2.15 *Let $f: \mathbb{C} \to \mathbb{C}$ be a bounded holomorphic function. Then f is constant.* □

Remarkably, we have an analogue of all of these results for Banach space-valued holomorphic functions. Formulating Cauchy's theorem or Cauchy's integral theorem in this setting would require defining integrals of Banach-space-valued functions. We will not do this here. However, the relevant version of Liouville's theorem is easily stated.

Theorem 2.16 *Let A be a complex Banach space. Let $f: \mathbb{C} \to A$ be a bounded holomorphic function. Then f is constant.* □

Chapter 3

Banach Algebras and Spectral Theory

3.1 Banach Algebras

Definition 3.1 *Let A be a vector space over the field $\mathbb{F}$. We call A an* algebra *if it is equipped with an associative* multiplication operation*, $A \times A \to A$, written $(x, y) \mapsto xy$ such that:*

- *Let $x, y, z \in A$. Then $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.*
- *Let $\alpha, \beta \in \mathbb{F}$ and $x, y \in A$. Then $(\alpha x)(\beta y) = (\alpha\beta)(xy)$.*

Definition 3.2 *We call a real or complex algebra, A, a* normed algebra *if it is also a normed vector space, and the norm $\| - \|$ satisfies the inequality*

$$\|xy\| \leq \|x\|\|y\|$$

for all $x, y \in A$. A normed algebra is called a Banach algebra *if it is complete.*

A normed or Banach algebra A is called unital *if there is an element $1 \in A$ such that $\|1\| = 1$, and $x1 = 1x = x$ for all $x \in A$. In this case we write $\lambda = \lambda 1 \in A$ for each scalar $\lambda \in \mathbb{C}$.*

Note that $\|0\| = 0$, so $1 \neq 0$, and the Banach algebra $\{0\}$ is not considered unital.

Example 3.3 *Let H be a Hilbert space. Let $\mathcal{L}(H)$ be the set of bounded linear operators from H to H. Then $\mathcal{L}(H)$ is a unital Banach algebra. Multiplication is defined by composition of operators. The norm is the operator norm.*

Example 3.4 *Let X be a compact metric space. Then the algebra $C(X)$, consisting of all continuous functions $X \to \mathbb{F}$ is a unital Banach algebra. Addition and multiplication of functions are defined pointwise. The norm is defined by the formula*

$$\|f\| = \sup\{|f(s)| \mid s \in X\}$$

The following result is an easy consequence of the definition.

Proposition 3.5 *In a normed algebra A the operations of addition and multiplication,*

$$A \times A \to A$$

are continuous. □

Corollary 3.6 *Let A be a normed algebra. Then the operations of addition and multiplication extend continuously to the completion, $\overline{A}$. The completion $\overline{A}$ is a Banach algebra.* □

This process of completion is useful for construction of examples. For most of our analysis to work, we really do need to work with Banach algebras rather than normed algebras.

Let us say an element x in a unital Banach algebra A is *invertible* if there is an element $x^{-1} \in A$ such that $x^{-1}x = xx^{-1} = 1$.

The following criterion for invertibility is called the *Carl Neumann criterion.*

Proposition 3.7 *Let A be a unital Banach algebra, and let $x \in A$. Suppose that $\|x\| < 1$. Then the element $1 - x$ is invertible, with inverse defined by the norm-convergent series*

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots.$$

Proof: Define

$$S_n = 1 + x + x^2 + \cdots + x^n$$

Then the sequence (S_n) is a Cauchy sequence in A, and therefore converges to some element $y \in A$. But

$$S_n(1-x) = (1-x) + (x - x^2) + \cdots + (x^n - x^{n+1}) = 1 - x^{n+1}$$

Hence

$$\|S_n(1-x) - 1\| = \| - x^{n+1}\| \leq \|x\|^{n+1}$$

so

$$\|y(1-x) - 1\| = \lim_{n\to\infty} \|S_n(1-x) - 1\| = 0$$

since $\|x\| < 1$. Therefore $y(1-x) = 1$. Similarly $(1-x)y = 1$. □

Note that completeness of the Banach algebra A is vital to the above proposition. The following is left as an exercise.

Corollary 3.8 *Let A be a unital Banach algebra, let $x \in A$, and $\lambda \in \mathbb{F}\backslash\{0\}$. Suppose that $\|x\| < |\lambda|$. Then the element $\lambda - x$ is invertible, with inverse defined by the norm-convergent series*

$$(\lambda - x)^{-1} = \lambda^{-1}(1 + \lambda^{-1}x + \lambda^{-2}x^2 + \cdots.$$

□

3.2 The Spectrum of an Element

Much of the theory of Banach algebras is based upon the following innocuous-seeming definition.

Definition 3.9 *Let A be a unital complex Banach algebra, and let $x \in A$. Then we define the* spectrum, *Spectrum*(x), *to be the set of complex numbers $\lambda \in \mathbb{C}$ such that the element $x - \lambda$ is not invertible.*

Note that all of our results about the spectrum are for *complex* Banach algebras. The notation $\sigma(x)$ is sometimes used to denote the spectrum, *Spectrum*(x).

Example 3.10 *Let $f \in C(X)$ be a continuous function. Since multiplication is defined pointwise, the spectrum Spectrum(f) is the image of the function f.*

Example 3.11 *Consider the Banach algebra $\mathcal{L}(\mathbb{C}^n)$ of all bounded linear operators on the space $\mathbb{C}^n$. Let $T \in \mathcal{L}(\mathbb{C}^n)$. Then T is an $n \times n$ complex-valued matrix.*

The matrix $T - \lambda I$ is not invertible if and only if $\det(T - \lambda I) = 0$, *that is to say that λ is an eigenvalue of T.*

So the spectrum, Spectrum(T), is the set of eigenvalues of T.

Thus the spectrum can be viewed as a generalisation of the set of eigenvalues of a matrix.

Theorem 3.12 *The spectrum Spectrum(x) is a compact subset of the set of complex numbers, $\mathbb{C}$, contained in the closed ball $\overline{B(0, \|x\|)}$.*

Proof: We begin by showing the spectrum is contained in the closed ball $\overline{B(0, \|x\|)}$. Let $\lambda \in \mathbb{C} \backslash \overline{B(0, \|x\|)}$. Then $\|x\| < |\lambda|$, so by corollary 3.8, the element $x - \lambda$ is invertible.

Hence $\lambda \notin$ *Spectrum*(x), and *Spectrum*$(x) \subseteq \overline{B(0, \|x\|)}$.

In particular the spectrum of x is a bounded subset of $\mathbb{C}$. To prove it is compact it now suffices, by the Heine-Borel theorem, to prove that it is closed.

We shall prove that the set

$$\mathbb{C} \backslash \mathit{Spectrum}(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \text{ is invertible}\}$$

is open.

So, let $\lambda \in \mathbb{C} \backslash$ *Spectrum*(x). Let $y = x - \lambda$. Then the element y is invertible. Set $\delta = 1/\|y^{-1}\|$. Choose $\mu \in \mathbb{C}$ such that $|\lambda - \mu| < \delta$.

Then

$$y^{-1}(x - \mu) = y^{-1}(y + \lambda - \mu) = 1 + (\lambda - \mu)y^{-1}$$

and $\|(\lambda - \mu)y^{-1}\| < 1$. So by the Carl Neumann criterion, the product $y^{-1}(x - \mu)$ is invertible. Since the element y^{-1} is certainly invertible, it follows that the element $x - \mu$ is invertible, and $\mu \in \mathbb{C} \backslash$ *Spectrum*(x).

Hence the set $\mathbb{C}\setminus Spectrum(x)$ is open, and we are done. □

We shall see in the next section that the spectrum of an element of a Banach algebra is always non-empty. This result needs more machinery than the above to prove.

Definition 3.13 *Let A be a unital Banach algebra, and let $x \in A$. Then we define the* spectral radius *of x*

$$R_\sigma(x) = \sup\{|\lambda| \mid \lambda \in Spectrum(x)\}.$$

By the above theorem, the spectral radius $R_\sigma(x)$ is well-defined, and $R_\sigma(x) \leq \|x\|$.

3.3 The Resolvent

Definition 3.14 *Let A be a unital Banach algebra, and let $x \in A$. Then we define the* resolvent set *of x to be the complement of the spectrum)*

$$\rho(x) = \mathbb{C}\setminus Spectrum(x).$$

In other words,

$$\rho(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \text{ is invertible}\}$$

We saw in the previous section that the resolvent set $\rho(x)$ is open.

Definition 3.15 *Let $\lambda \in \rho(x)$. Then we define the* resolvent

$$R_\lambda(x) = (\lambda - x)^{-1}.$$

The following result is left as an exercise.

Proposition 3.16 *Let $x \in A$. Then $\|R_\lambda(x)\| \to 0$ as $\lambda \to \infty$.* □

The following equation is called the *resolvent identity.*

Lemma 3.17 *Let $\lambda, \mu \in \rho(x)$. Then*

$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu.$$

Proof: Certainly

$$(\mu - x) - (\lambda - x) = \mu - \lambda.$$

Hence

$$R_\lambda - R_\mu = R_\lambda((\mu - x) - (\lambda - x))R_\mu = (\mu - \lambda)R_\lambda R_\mu$$

and we are done. □

From the above it follows that $R_\lambda(x)R_\mu(x) = R_\mu(x)R_\lambda(x)$.

Lemma 3.18 *We have a holomorphic function $f\colon \rho(x) \to A$ defined by the formula $f(\lambda) = R_\lambda(x)$.*

Proof: Let $g(\lambda) = -R_\lambda(x)^2 = -(\lambda - x)^{-2}$. Then

$$\left\| \frac{f(\lambda + h) - f(\lambda)}{h} - g(\lambda) \right\| = \left\| \frac{1}{h}((\lambda - x + h)^{-1} - (\lambda - x)^{-1}) + (\lambda - x)^{-2} \right\|.$$

Now, for h sufficiently small, by the Carl Neumann criterion, $\lambda - x + h$ is invertible (as the above requires), and

$$(\lambda - x + h)^{-1} = (\lambda - x)^{-1}(1 - h(\lambda - x)^{-1} + h^2(\lambda - x)^{-2} - \cdots).$$

So a

$$\left\| \frac{f(\lambda + h) - f(\lambda)}{h} - g(x) \right\| = \left\| h(\lambda - x)^{-3} - h^2(\lambda - x)^{-4} + \cdots \right\|.$$

We see

$$\left\| \frac{f(\lambda + h) - f(\lambda)}{h} - g(x) \right\| \to 0$$

as $h \to 0$. Hence the function $f(\lambda)$ is holomorphic, with derivative $g(\lambda)$. □

Our next result is a far-reaching generalisation of the fact that any complex matrix has at least one eigenvector.

Theorem 3.19 *Let A be a unital Banach algebra, and let $x \in A$. Then Spectrum$(x) \neq \emptyset$.*

Proof: Suppose *Spectrum*$(x) = \emptyset$. Then the resolvent set, $\rho(x)$, is all of $\mathbb{C}$, and by the above we have a holomorphic function $f\colon \mathbb{C} \to A$ given by the formula $f(\lambda) = (x - \lambda)^{-1}$.

By proposition 3.16, we know that $\|f(\lambda)\| \to 0$ as $\lambda \to \infty$. It follows by Liouville's theorem that $f(\lambda) = 0$ for all λ, that is to say $(x - \lambda)^{-1} = 0$ for all $\lambda \in \mathbb{C}$. But this is clearly impossible.

Therefore *Spectrum*$(x) \neq \emptyset$, as required. □

We conclude this chapter with a generalisation of the Cayley-Hamilton theorem from linear algebra called the *spectral mapping theorem for polynomials*.

Theorem 3.20 *Let A be a unital Banach algebra, and let $x \in A$. Let p be a polynomial. Then*

$$\mathit{Spectrum}[p(x)] = p[\mathit{Spectrum}(x)].$$

Proof: Without loss of generality, let p have degree $n \geq 1$ and leading coefficient 1. Let $\mu \in \mathbb{C}$. Then by the fundamental theorem of algebra, we can write

$$p(x) - \mu = (x - \beta_1)(x - \beta_2) \cdots (x - \beta_n)$$

where $\beta_1, \ldots, \beta_n$ are the zeros of the polynomial $p(\lambda)$.

Let $\mu \in Spectrum[p(x)]$. Suppose $\beta_i \in \rho(x)$ for all i. Then each term $(x - \beta_i)$ is invertible, meaning $p(x) - \mu$ is invertible, and $\mu \in \rho[p(x)]$, which is a contradiction.

Thus $(x - \beta_i)$ is not invertible for some i, meaning $\beta_i \in Spectrum(x)$. But $p(\beta_i) = \mu$, so $\mu \in p[Spectrum(x)]$. We thus have

$$Spectrum[p(x)] \subseteq p[Spectrum(x)].$$

Conversely, let $\mu \in \rho[p(x)]$. The factors $(x - \beta_i)$ commute with each-other, so

$$R_\mu(p(x))(x - \beta_1) \cdots (x - \beta_n) = 1$$

and

$$(x - \beta_n)(x - \beta_1) \cdots (x - \beta_{n-1}) R_\mu(p(x)) = 1.$$

It follows that the element $x - \beta_n$ is invertible, so $\beta_n \in \rho(x)$. As before, $\mu = p(\beta_n)$, so $\mu \in p[\rho(x)]$. Thus $\rho[p(x)] \subseteq p[\rho(x)]$, which means

$$p[Spectrum(x)] \subseteq Spectrum[p(x)].$$

This completes the proof. □

Actually, one can generalise the result from polynomials to any holomorphic function defined on the neighbourhood of $Spectrum(x)$, but the proof of such a result is beyond the scope of this course.

Chapter 4

The Spectral Theory of Operators on Hilbert Space

4.1 Unitary Operators

Throughout this chapter we consider the complex Banach algebra $\mathcal{L}(H)$ of bounded linear maps from H to H, where H is a Hilbert space. This Banach algebra is unital, with unit the identity map $I\colon H \to H$.

As well as addition and multiplication, we have another operation called *involution* defined by taking the adjoint of a bounded linear map T.

The following properties hold for all $S, T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$.

- $(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*$.
- $(ST)^* = T^* S^*$.
- $(S^*)^* = S$.
- $\|S^* S\| = \|S\|^2$.

We have already seen the first three of these properties. The last one is called the C^**-identity*, and is left as an exercise.

Recall that we call $U \in \mathcal{L}(H)$ *a unitary operator* (or just a unitary) if $UU^* = U^*U = I$.

Thus U is unitary if it is invertible, and $U^* = U^{-1}$.

Example 4.1 *Let $H = \mathbb{C}^2$. Then $\mathcal{L}(H)$ is the set of 2×2 complex matrices, and the multiplication in $\mathcal{L}(H)$ is matrix multiplication.*

Let

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$U^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Observe that $U^*U = UU^* = I$*, so* U *is unitary.*

The proof of the following results are left as exercises.

Proposition 4.2 *Let* $U \in \mathcal{L}(H)$ *be unitary. Then:*

- *The map* U *is an isometry.*
- $\|U\| = 1$.
- U^* *is unitary.*

□

Proposition 4.3 *The product of two unitary operators is unitary.* □

Theorem 4.4 *Let* $U \in \mathcal{L}(H)$ *be unitary. Then*

$$Spectrum(U) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}.$$

Proof: By the above, $\|U\| = \|U^*\| = 1$.

Let $\lambda \in \mathbb{C}$, $|\lambda| > 1$. Then by the Carl Neumann criterion, the map $I - \lambda^{-1}U$ is invertible in $\mathcal{L}(H)$, that is to say $\lambda I - U$ is invertible, and $\lambda \notin Spectrum(U)$.

Let $\lambda \in \mathbb{C}$, $|\lambda| < 1$. Then by the Carl Neumann criterion, the map $I - \lambda U^*$ is invertible in $\mathcal{L}(H)$. Multiplying by $-U$, we see that map $\lambda I - U$ is invertible, and again $\lambda \notin Spectrum(U)$.

Thus if $\lambda \in Spectrum(U)$, then $|\lambda| = 1$. The result now follows. □

4.2 Self-Adjoint Operators

Recall that we call a map a $T \in \mathcal{L}(H)$ a *self-adjoint operator* (or just self-adjoint) if $T = T^*$.

Now, and element $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbb{C}$, if we have a corresponding *eigenvector* $v \in H$, where $v \neq 0$, and $T(v) = \lambda v$, that is to say $(\lambda I - T)(v) = 0$. Thus, $\lambda \in \mathbb{C}$ is an eigenvalue if and only if the linear map $\lambda I - T$ is not injective.

So, if $\lambda \in \mathbb{C}$ is an eigenvalue of T, then $\lambda I - T$ has no inverse in the algebra $\mathcal{L}(H)$, so $\lambda \in Spectrum(T)$.

Example 4.5 *Consider the operator* $S\colon l^2 \to l^2$ *defined by the formula*

$$S(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots).$$

Then S *is injective, so* 0 *is not an eigenvalue of* S*. However,* S *is not surjective, so it is not invertible; therefore,* $0I - S$ *is not invertible, and* $0 \in Spectrum(S)$*.*

Of course the operator S is not self-adjoint. Nevertheless, there are plenty of self-adjoint operators where there are values in the spectrum that are not eigenvalues; see the exercises.

Proposition 4.6 *Let $T \in \mathcal{L}(H)$ be self-adjoint. Let λ be an eigenvalue of T. Then $\lambda \in \mathbb{R}$.*

Furtherm eigenvectors corresponding to different eigenvalues are orthogonal.

Proof: Let $\lambda \in \mathbb{C}$ be an eigenvalue, and let v be a corresponding eigenvector. Then

$$\lambda\langle v, v\rangle = \langle v, Tv\rangle = \langle Tv, v\rangle = \overline{\lambda}\langle v, v\rangle.$$

Since $v \neq 0$, $\langle v, v\rangle \neq 0$, so $\lambda = \overline{\lambda}$ and $\lambda \in \mathbb{R}$.

Now, let λ and μ be eigenvalues, with $\lambda \neq \mu$. Let v and w be corresponding eigenvectors. Certainly $\lambda \in \mathbb{R}$, so

$$\lambda\langle v, w\rangle = \langle \lambda v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \mu\langle v, w\rangle.$$

So $(\lambda - \mu)\langle v, w\rangle = 0$, and $\langle v, w\rangle = 0$ as desired. □

Actually, it turns out we can go further than the above with the following theorem. We omit the proof.

Theorem 4.7 *Let $T \in \mathcal{L}(H)$ be self-adjoint. Then $Spectrum(T) \subseteq \mathbb{R}$.*

Further, if we set

$$m(T) = \inf_{\|v\|=1} \langle v, Tv\rangle \qquad M(T) = \sup_{\|v\|=1} \langle v, Tv\rangle.$$

then $m(T), M(T) \in Spectrum(T)$, and $Spectrum(T) \subseteq [m(T), M(T)]$. □

Recall that we can define the *spectral radius* of an operator T:

$$R_\sigma(T) = \sup\{|\lambda| \mid \lambda \in Spectrum(T)\}$$

and we have $R_\sigma(T) \leq \|T\|$. The above result tells us that

$$R_\sigma(T) = \max(|m(T)|, |M(T)|)$$

when $T \in \mathcal{L}(H)$ is self-adjoint.

Corollary 4.8 *Let $T \in \mathcal{L}(H)$ be self-adjoint. Then*

$$\|T\| = R_\sigma(T) = \max(|m(T)|, M(T)|).$$

Proof: Let $\mu \in \mathbb{R}$ be such that $|\mu| > R_\sigma(T)$. Then $\mu, -\mu \notin Spectrum(T)$, so the operator

$$(T^2 - \mu^2 I) = (T + \mu I)(T - \mu I)$$

is invertible, with bounded inverse.

We know that $R_\sigma(T) \leq \|T\|$. Suppose $R_\sigma(T) > \|T\|$. Then we can apply the above to the real number $\mu = \|T\|$ to conclude that the operator $T^2 - \mu^2 I$ is invertible, so $\mu^2 = \|T\|^2 \notin Spectrum(T^2)$.

But

$$M(T^2) = \sup_{\|v\|\leq 1} \langle v, T^2 v\rangle = \sup_{\|v\|\leq 1} \langle Tv, Tv\rangle = \|T\|^2.$$

So by the above, $\|T\|^2 = M(T^2) \in Spectrum(T^2)$, which is a contradiction. We conclude $\|T\| = R_\sigma(T)$, and we are done. □

4.3 Compact Operators

Recall that we call a subset K of a metric space X *compact* if every sequence in K has a convergent subsequence, with limit also in K.

It is a fact that every compact subset of a metric space is closed and bounded. The converse is not true in general. However, every closed subset of a compact metric space is compact. A more sophisticated (but very useful) result is the Heine-Borel theorem, which tells us that any closed bounded subset of a *finite-dimensional* normed vector space is compact.

Definition 4.9 *Let V and W be normed vector spaces. A linear map $T\colon V \to W$ is said to be* compact *if the closure of the image of the unit ball, $\overline{T[B(0,1)_V]} \subseteq W$, is compact.*

Example 4.10 *Let H be an infinite-dimensional Hilbert space. Let $I\colon H \to H$ be the identity map. Then $\overline{I[B(0,1)_H]} = \overline{B(0,1)_H} = \{v \in H \mid \|v\| \leq 1\}$, which is not compact since H is infinite-dimensional.*

To see this set is not compact, observe that we can choose an infinite orthonormal sequence $(e_n)_{n\in\mathbb{N}}$. This sequence lies within the set $\overline{B(0,1)_H}$, but has no convergent subsequence.

Hence the operator $I\colon H \to H$ is not compact.

Example 4.11 *Let $T\colon V \to W$ be a bounded linear map such that the image $T[V]$ is finite-dimensional.*

Then $T[V]$ will be a closed subspace of W, so $\overline{T[B(0,1)_V]} \subseteq T[V]$

Since T is bounded, if $v \in \underline{B(0,1)_V}$, then $\|Tv\| \leq \|T\| \cdot \|v\| < \|T\|$. Hence $\|w\| \leq \|T\|$ whenever $w \in \overline{T[B(0,1)_V]}$. Thus the set $\overline{T[B(0,1)_V]}$ is a closed bounded subset of a finite-dimensional normed vector space, so it is compact by the Heine-Borel theorem. It follows that the operator T is compact.

We can reformulate the definition as follows.

Proposition 4.12 *Let $T\colon V \to W$ be a linear map. Then T is compact if and only if for any bounded set $B \subseteq V$, the closure of the image $\overline{T[B]} \subseteq W$ is compact.*

Proof: Suppose that whenever $B \subseteq V$ is bounded, the closure of the image $\overline{T[B]} \subseteq W$ is compact. Then taking $B = B(0,1)_V$, we see that T is compact.

Conversely, let $T\colon V \to W$ be compact. Let $B \subseteq V$ be bounded. Then we have $c > 0$ such that $B \subseteq B(0,c)_V$. Hence

$$\overline{T[B]} \subseteq \overline{T[B(0,c)_V]} = c\overline{T[B(0,1)_V]}$$

and the space on the right is compact as T is compact.

Hence the set $\overline{T[B]}$ is a closed subset of a compact metric space, and is therefore itself compact. □

The following is a sort of converse of example 4.11.

Corollary 4.13 *Let $T: V \to W$ be a linear map. Let T be compact. Then T is also bounded.*

Proof: The sphere
$$S = \{v \in V \mid \|v\| = 1\}$$
is certainly bounded, meaning the closure $\overline{T[S]}$ is compact, so the image $T[S]$ is bounded.

Hence we have $M \geq 0$ such that $\|T(v)\| \leq M$ whenever $v \in V$ with $\|v\| = 1$. The result now follows. □

The proof of the following is left as an exercise.

Lemma 4.14 *Let $T: V \to W$ be a linear map between normed vector spaces. Then T is compact if and only if for every bounded sequence (x_n) in V, the image (Tx_n) has a convergent subsequence.* □

Theorem 4.15 *Let V be a normed vector space, and let W be a Banach space. Let $T: V \to W$ be a bounded linear map. Suppose we have a sequence, (T_n), of compact operators $T_n: V \to W$ such that $\|T - T_n\| \to 0$ as $n \to \infty$. Then T is a compact operator.*

Proof: This proof is an example of a 'diagonal argument'. Here goes...

Let (x_n) be a bounded sequence in V. Then, since T_1 is compact, by the above lemma the sequence (x_n) has a subsequence $(x_{1,m})$ such that $(Tx_{1,m})$ converges. In particular, the sequence $(T_1 x_{1,m})$ is Cauchy.

Since T_2 is compact, by the above lemma the sequence $(x_{1,m})$ has a subsequence $(x_{2,m})$ such that $(T_2 x_{2,m})$ converges. In particular, the sequence $(T_2 x_{2,m})$ is Cauchy.

Repeating this argument (or, more formally, using induction) we get for each k a subsequence $(x_{k,m})_{m\in\mathbb{N}}$ of (x_n) such that:

- $(x_{k+1,m})_{m\in\mathbb{N}}$ is a subsequence of (x_n).
- $(T_k x_{k,m})_{m\in\mathbb{N}}$ is Cauchy,

Let $y_m = x_{m,m}$ (this is the 'diagonal' bit of the proof). Then (y_m) is a subsequence of (x_n), and for every k the sequence $(T_k y_m)_{m\in\mathbb{N}}$ is Cauchy.

Now the sequence (x_n) is bounded. Fix $C > 0$ such that $\|x_n\| \leq C$ for all $n \in \mathbb{N}$. Then certainly $\|y_m\| \leq C$ for all m.

Let $\varepsilon > 0$. Then we have $p \in \mathbb{N}$ such that $\|T - T_p\| < \frac{\varepsilon}{3C}$, and we have $N \in \mathbb{N}$ such that $\|T_p y_j - T_p y_k\| < \frac{\varepsilon}{3}$ whenever $j, k \geq N$.

Now, let $j, k \geq N$. Then
$$\|Ty_j - Ty_k\| \leq \|Ty_j - T_p y_j\| + \|T_p y_j - T_p y_k\| + \|T_p y_k - Ty_k\|.$$

But
$$\|Ty_j - T_p y_j\| \leq \|T - T_p\| \cdot \|y_j\| \leq C\|T - T_p\| < \frac{\varepsilon}{3}.$$

Similarly, $\|T_p y_k - T y_k\| < \frac{\varepsilon}{3}$, and we know $\|T_p y_j - T_p y_k\| < \frac{\varepsilon}{3}$. Thus, in summary, if $j, k \geq N$, then $\|Ty_j - Ty_K\| < \varepsilon$. So the sequence (Ty_m) is Cauchy, and therefore converges since W is a Banach space.

Thus, given a bounded sequence (x_n), we have constructed a subsequence (y_m) such that (Ty_m) converges. By the above lemma, this means that the linear map T is a compact operator. □

Combining the above with example 4.11 yields the following.

Corollary 4.16 *Let V be a normed vector space, and let W be a Banach space. Let $T: V \to W$ be a bounded linear map. Suppose we have a sequence, (T_n), of operators $T_n: V \to W$ with finite-dimensional images, such that $\|T - T_n\| \to 0$ as $n \to \infty$. Then T is a compact operator.* □

Actually, the above corollary has a converse, commonly called the *approximation property.* This converse is only valid for operators on Hilbert spaces, and the proof of is beyond the scope of this course.

Theorem 4.17 *Let H be a Hilbert space. Let $K: H \to H$ be a compact operator. Then there is a sequence, (T_n), of operators $T_n: H \to H$ with finite-dimensional images, such that $\|T - T_n\| \to 0$ as $n \to \infty$.* □

The following is an exercise, using theorem 4.15.

Example 4.18 *Define $T: l^1 \to l^1$ by*

$$T(a_1, a_2, a_3, \ldots) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, \ldots).$$

Then T is a compact operator.

We conclude our introduction to compact operators with the following result.

Proposition 4.19 *Let V be a normed vector space. Let $T: V \to V$ be a bounded linear map, and let $K: V \to V$ be a compact operator. Then the composites $T \circ K$ and $K \circ T$ are compact.*

Proof: Since T is a bouned linear map, the image

$$T[B(0,1)] = \{Tx \mid \|x\| < 1\}$$

is bounded. Hence, by proposition 4.12, the set $\overline{K[T[B(0,1)]]} = \overline{K \circ T[B(0,1)]}$ is compact, meaning the operator $K \circ T$ is compact.

As for the other composite, let (v_n) be a bounded sequence in V. By lemma 4.14, there is a subsequence (v_{n_k}) such that (Kv_{n_k}) converges. Since the map T is bounded, it is continuous, and therefore takes convergent sequences to convergent sequences. So the sequence (TKv_{n_k}) converges. Again applying lemma 4.14, we see that the composite $T \circ K$ is compact, and we are done. □

The proof of the following result is left as an exercise.

Proposition 4.20 *Let H be a Hilbert space, and let (e_n) be an orthonormal sequence in H. Let (λ_n) be a sequence in $\mathbb{C}$ such that $\lambda_n \to 0$ as $n \to \infty$. Then we can define a compact operator $K\colon H \to H$ by writing*

$$K(v) = \sum_{n=1}^{\infty} \lambda_n \langle e_n, v \rangle e_n$$

whenever $v \in H$. Further, $\|T\| = \max_{n\in\mathbb{N}} |\lambda_n|$. □

Now, remarkable, for compact and self-adjoint operators, we basically have a converse to the above; a result telling us that if H is a Hilbert space, and $K\colon H \to H$ is a compact and self-adjoint operator, then H has a basis made up of eigenvectors of H.

More precisely, the following result, known as the *spectral theorem for compact self-adjoint operators*, holds. We omit the proof.

Theorem 4.21 *The $K\colon H \to H$ be a compact self-adjoint operator. Then:*

- *The Hilbert space H has a basis made up of eigenvectors of K. This basis contains a countable or finite basis of the space $(\ker K)^\perp$.*
- *If the orthonormal basis of the space $(\ker K)^\perp$ is finite, let use write it $\{e_1, \ldots e_N\}$. Let $\lambda_1, \ldots, \lambda_N$ be the corresponding eigenvalues. Then*

  $$Kv = \sum_{n=1}^{n} \lambda_n \langle e_n, v \rangle e_n$$

  for all $v \in H$.
- *If the orthonormal basis of the space $(\ker K)^\perp$ is infinite, we can order it so that it is a sequence (e_n), and the corresponding sequence of eigenvalues, (λ_n) is such that the sequence $(|\lambda_n|)$ is monotonic decreasing and converges to 0. Further, we can write*

  $$Kv = \sum_{n=1}^{\infty} \lambda_n \langle e_n, v \rangle e_n$$

  for all $v \in H$.

□

4.4 Integral Kernels

Let $C([0,1] \times [0,1])$ be the space of continuous functions $[0,1] \times [0,1] \to \mathbb{C}$. Then we can define an inner product on $C([0,1] \times [0,1])$ by the formula

$$\langle g, h \rangle = \int_0^1 \int_0^1 \overline{g(s,t)} h(s,t)\ ds\ dt.$$

We can complete $C([0,1]\times[0,1])$ with respect to the norm, $\|-\|_2$, induced by this inner product to obtain a Hilbert space $L^2([0,1]\times[0,1])$. The details are left as an exercise, as is the proof of the following propostion.

Lemma 4.22 *Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $L^2[0,1]$, where $e_n \in C[0,1]$ for all n. For $m,n\in\mathbb{N}$, define a function $\phi_{m,n}\in C([0,1]\times[0,1])$ by the formula $\phi(s,t) = e_m(s)e_n(t)$. Then $(\phi_{m,n})_{m,n\in\mathbb{N}}$ is a basis for $L^2([0,1]\times[0,1])$.* □

Theorem 4.23 *Let $k\in C([0,1]\times[0,1])$. Then we have a compact operator $K\colon L^2[0,1]\to L^2[0,1]$ defined by the formula*

$$(Kf)(s) = \int_0^1 k(s,t)f(t)\ dt$$

for $f\in C[0,1]$.

Further, $\|K\|\leq\|k\|_2$.

Proof: Observe that the above formula defines a linear map $K\colon C[0,1]\to C[0,1]$. If $f\in C[0,1]$, then

$$\|Kf\|^2 = \int_0^1 \left|\int_0^1 k(s,t)f(t)\ dt\right|^2\ ds$$

so by the Cauchy-Schwarz inequality on the space $L^2[0,1]$

$$\|Kf\|^2 \leq \int_0^1 \left(\int_0^1 |\overline{k(s,t)}|^2\ dt\right)\left(|f(t)|^2\ dt\right)\ ds.$$

Hence $\|Kf\|^2\leq(\|k\|_2)^2\|f\|^2$. So the linear map K is bounded, with norm $\|K\|\leq\|k\|_2$. Hence the map K extends uniquely to a bounded linear map $K\colon L^2[0,1]\to L^2[0,1]$, and the inequality $\|K\|\leq\|k\|_2$ still holds.

Now, let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for $L^2[0,1]$, and let $(\phi_{m,n})_{m,n\in\mathbb{N}}$ be the associated orthonormal basis for $L^2([0,1]\times[0,1])$ as in the above lemma. Set

$$k_N = \sum_{m,n=1}^{N} \langle\phi_{m,n},k\rangle\phi_{m,n}$$

Then $\|k_N - k\|_2\to 0$ as $N\to\infty$. Define

$$(K_nf)(s) = \int_0^1 k_N(s,t)f(t)\ dt.$$

Then by the above,

$$\|K_N - K\|\leq\|k_N-k\|_2\to 0 \text{ as } N\to\infty.$$

Observe

$$(K_N f)(s) = \sum_{m,n=1}^{N} \int_0^1 \langle \phi_{m,n}, k \rangle e_m(s) e_n(t) f(t)\ dt$$

so

$$K_N f = \sum_{m,n=1}^{N} \langle \phi_{m,n}, k \rangle \langle \overline{e_n}, f \rangle e_m.$$

We see that the operator K_N has finite-dimensional image, and the operator K is a norm-limit of the operators K_N. Thus the operator K is compact by theorem 4.15. □

We call the above map $k\colon [0,1] \times [0,1] \to \mathbb{C}$ the *integral kernel* of the operator K. The following is easy to see.

Proposition 4.24 *Suppose $k(s,t) = \overline{k(t,s)}$ for all $s, t \in [0,1]$. Then the operator K is self-adjoint.* □

Let us look at an extended example.

Proposition 4.25 *Let $f\colon [0,1] \to \mathbb{C}$ be a continuous function. Then the following are equivalent for a continuous function $y\colon [0,1] \to \mathbb{C}$:*

- *y satisfies the differential equation*

$$y''(x) + f(x) = 0$$

  with boundary conditions

$$y(0) = y(1) = 0.$$

- *$y = Kf$, where $K\colon L^2[0,1] \to L^2[0,1]$ is the compact operator with integral kernal defined by the formula*

$$k(s,t) = \begin{cases} t(1-x) & 0 \leq t \leq x \\ x(1-t) & x \leq t \leq 1 \end{cases}$$

□

The proof of this result is purely computational, and left as an exercise.

Corollary 4.26 *A number $\lambda \in \mathbb{C}$ is an eigenvalue of the above operator K, with eigenvector y if and only if*

$$y'' + \frac{1}{\lambda} y = 0$$

and $y(0) = y(1) = 0$.

Proof: By the above result, if $Kf = 0$ then $f = 0$, so 0 is not an eigenvalue of K. So, let $\lambda \neq 0$, and $f(x) = \frac{1}{\lambda}y$. Then, again by the above, the stated differential equation holds if and only if $y = K(\frac{1}{\lambda y})$, that is to say λ is an eigenvalue of K with eigenvector y. □

Now, since the operator K is, by definition, self-adjoint, every eigenvalue of K must be real. So, let $\lambda \in \mathbb{R}\backslash\{0\}$.

Then the equation

$$y'' + \frac{1}{\lambda}y = 0$$

is a second order differential equation with constant coefficients. Let $\lambda < 0$. Write $\lambda = -\frac{1}{a^2}$. Then we are looking at the differential equation

$$y'' = a^2 y$$

which has general solution

$$y(x) = Ae^{ax} + Be^{-ax}$$

where $A, B \in \mathbb{C}$ are constants. The boundary conditions $y(0) =$ and $y(1) = 0$ tell us that $y = 0$, so in this case the only solution is trivial, and λ is not an eigenvalue.

Let $\lambda > 0$. Write $\lambda = \frac{1}{a^2}$. Then we are looking at the differential equation

$$y'' = -a^2 y$$

which has general solution

$$y(x) = A\cos(ax) + B\sin(ax)$$

The boundary condition $y(0) = 0$ immediately tells us $A = 0$. The condition $y(1) = 0$ tells us $B\sin(a) = 0$. Thus we have the possibility of a non-zero solution if $\sin(a) = 0$, that is $a = n\pi$, where $n \in \mathbb{N}$.

Hence the eigenvalues of the operator K are given by

$$\lambda_n = \frac{1}{n^2\pi^2}.$$

and we have corresponding eigenvectors

$$y_n(x) = \sin(n\pi x).$$

Observe

$$\|y_n\|^2 = \int_0^1 \sin^2(n\pi x)\ dx = \frac{1}{2}$$

so $\|y_n\| = \frac{1}{\sqrt{2}}$.

So set

$$e_n(x) = \sqrt{2}\sin(n\pi x).$$

Then by the spectral theorem for compact self-adjoint opertors, the sequence $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space $L^2[0,1]$.

Chapter 5

Fredholm Operators

5.1 The Notion of a Fredholm Operator

Definition 5.1 *Let H be a Hilbert space. A bounded linear operator $T: H \to H$ is called* Fredholm *if the image $im(T)$ is closed, and the kernels* $\ker T$ *and* $\ker T^*$ *are finite-dimensional.*

If $T: H \to H$ is a Fredholm operator, we define the index *of T to be*

$$Ind(T) = \dim \ker T - \dim \ker T^*.$$

Example 5.2 *Recall we define the right-shift and left-shift operators $R, L: l^2 \to l^2$ by the formulae*

$$R(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots)$$

and

$$L(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$$

respectively. We have $R^* = L$. Clearly R has a closed image.

Observe R is injective, so $\ker R = \{0\}$*, and*

$$\ker L = \{(a, 0, 0, \ldots) \mid a \in \mathbb{C}\}$$

so $\dim \ker L = 1$*. It follows that R is Fredholm, and $Ind(R) = -1$. Similarly, L is Fredholm and $Ind(L) = 1$.*

As a more trivial example, any Fredholm operator on a finite-dimensional Hilbert space has index zero; the proof is left as an exercise. More clearly, any invertible bounded linear map is Fredholm, with index zero.

The following fundamental result on Fredholm operators is called *Atkinson's theorem*

Theorem 5.3 *Let H be a Hilbert space, and let $T: H \to H$ be a bounded linear map. Then the following are equivalent:*

- *T is a Fredholm operator.*
- *There is a bounded linear map $S\colon H \to H$ such that $ST - I$ and $TS - I$ are compact operators.*

Proof: Before we begin the proof, recall that if H is a Hilbert space, $V \subseteq H$ is a closed subspace, and $W \subseteq H$ is a finite-dimensional subspace, then $V + W$ is a closed subspace of H.

Anyway, suppose that T is a Fredholm operator. Observe that the restriction $T_0\colon (\ker T)^\perp \to T[H]$ is injective. It is clearly also surjective. So the map T_0 is bijective, and as $T[H]$ is closed, we can deduce by the open mapping theorem that there is a bounded linear map $S_0\colon T[H] \to (\ker T)^\perp$ that is the inverse of T_0.

Again, because the image $T[H]$ is closed, we can write $H = T[H] \oplus T[H]^\perp$. Let us extend S_0 to a bounded linear map $S\colon H \to H$ by setting $S(v) = 0$ whenever $v \in T[H]^\perp$. Let $A = ST - I$ and $B = TS - I$. We must show that A and B are compact.

Observe that A is a bounded linear map, and

$$Av = \begin{cases} -v & v \in \ker T \\ 0 & v \in (\ker T)^\perp \end{cases}$$

But we know that $\ker T$ is finite-dimensional, so the operator A has finite-dimensional image, and so must be compact.

Observe that B is a bounded linear map, and

$$Bv = \begin{cases} 0 & v \in T[H] \\ -v & v \in T[H]^\perp \end{cases}$$

But $T[H]^\perp = \ker T^*$, so $T[H]^\perp$ is finite-dimensional, and the operator B also has finite-dimensional image, and is therefore compact.

We conclude that $A = ST - I$ and $B = TS - I$ are both compact.

Conversely, suppose that $A = ST - I$ and $B = TS - I$ are both compact. Let $v \in \ker T$. Then $Av = -v$, so v is a (-1)-eigenvector of A. By lemma **??**, since A is compact, the space of (-1)-eigenvectors is finite-dimensional, and hence $\ker T$ is finite-dimensional. Similarly, $\ker T^*$ is finite-dimensional.

It remains to prove that the image, $T[H]$, is closed. By the approximation property (theorem 4.17), we have an operator C with finite-dimensional image such that $\|C - A\| < \frac{1}{2}$.

Let $v \in \ker C$. Then

$$\|S\|\|Tv\| \geq \|STv\| = \|v + Av\| = \|v + Av - Cv\| \geq \|v\| - \|A - C\|\|v\| > \frac{1}{2}\|v\|.$$

We claim now that $T[\ker C]$ is closed. Let $y \in \overline{T[\ker C]}$. Let (y_n) be a sequence in $T[\ker C]$ such that $\|y_n - y\| \to 0$ as $n \to \infty$. Then (y_n) is a Cauchy sequence in $T[\ker C]$, so we have a sequence (x_n) in $\ker C$ such that $Tx_n = y_n$ for all n.

The above inequality tells us that the sequence (x_n) is also Cauchy. Since $\ker C$ is closed, the sequence (x_n) converges in norm to some limit $x \in \ker C$. By continuity of the map T, $Tx = y$, so $y \in T[\ker C]$. This establishes that $T[\ker C]$ is closed.

Now we have $H = \ker C \oplus (\ker C)^\perp$, and $(\ker C)^\perp = C^*[H]$, which is finite-dimensional since $C[H]$ is finite-dimensional. Therefore

$$T[H] = T[\ker C] \oplus T[(\ker C)^\perp.$$

The first of these subspaces is closed, and the second of these spaces is finite-dimensional. The sum of a closed and a finite-dimensional subspace is closed. So the image $T[H]$ is closed, and the operator T is Fredholm. □

The above theorem provides a handy way of checking when an operator T is Fredholm; we try to find an "inverse modulo compact operators", S, that is to say a bounded linear map S such that $ST - I$ and $TS - I$ are compact. Such an operator S is called a *parametrix* for T.

Corollary 5.4 *Let $T\colon H \to H$ be a Fredholm operator. Then the adjoint, T^*, is also Fredholm, and $Ind(T^*) = -Ind(T)$.*

Proof: This follows immediately from the above, together with the fact that the adjoint of a compact operator is compact. □

Note that this statement is not as obvious as it may at first appear, as it is not immediately apparent (though, by the above, it is true) that the adjoint of a Fredholm operator has to have closed image.

The next result needs the notion of quotient spaces. Specifically, we have the following. The details are left as an exercise.

Proposition 5.5 *Let V be a Banach space, and let W be a closed subspace. Let V/W be the quotient vector space, with quotient map $\pi\colon V \to V/W$. Then the quotient space V/W is a Banach space, with norm*

$$\|\pi(v)\| = \inf\{\|w\| \mid w \in \pi(v)\}.$$

Further, if $V = W \oplus X$ for a Banach space X, then the spaces X and V/W are isomorphic. □

Proposition 5.6 *Let $T\colon H \to H$ be a Fredholm operator. Then the spaces $\ker T$ and $H/\ker T$ are finite-dimensional, and $Ind(T) = \dim\ker T - \dim(H/\,im\,T)$.*

Proof: It suffices to show that the spaces $H/\,im\,T$ and $\ker T^*$ are isomorphic. Because the operator T is Fredholm, the image, $im\,T$, is closed. Let $v \in im\,T^\perp$. Then for any $w \in H$, we have that $\langle v, Tw\rangle = 0$, and so $\langle T^*v, w\rangle = 0$. It follows thar $T^*v = 0$. Similarly, if $T^*v = 0$ then $v \in (im\,T)^\perp$.

So $(im\,T)^{\perp} = \ker T^{*}$. By the Hilbert space projection theorem, $H = im\,T \oplus \ker T^{*}$. It now follows by the above that the spaces $\ker T^{*}$ and H/imT are isomorphic, as required. □

5.2 Properties of the Fredholm Index

Before looking at some properties of the Fredholm index, we need a small amount of machinery involving exact sequences.

Definition 5.7 *Let $V_0, V_1, \ldots, V_{n+1}$ be vector spaces, and let $T_k\colon V_k \to V_{k+1}$ be linear maps. Then the sequence*

$$V_0 \stackrel{T_0}{\to} V_1 \stackrel{T_1}{\to} V_2 \to \cdots \stackrel{T_n}{\to} V_{n+1}$$

of vector spaces and linear maps is called exact *if $im\,T_k = \ker T_{k+1}$ for all k.*

Lemma 5.8 *Let*

$$0 \stackrel{0}{\to} V_1 \stackrel{T_1}{\to} V_2 \to \cdots \to V_n \stackrel{0}{\to} 0$$

be an exact sequence of finite-diminesional vector spaces.

Then

$$\sum_{k=1}^{n} (-1)^k \dim V_k = 0.$$

Proof: For each k, write $V_k = N_k \oplus W_k$, where $N_k = \ker T_k$. The sequence being exact and everything being finite-dimensional tells us that the map $T_k|_{W_k}\colon W_k \to N_{k+1}$ is an isomorphism. Hence $\dim W_k = \dim N_{k+1}$ for each k. So for each k,

$$\dim V_k = \dim N_k + \dim W_k = \dim N_k + \dim N_{k+1}.$$

Since the first an last terms in the sequence are zero, we have that $\dim N_1 = 0$ and $\dim N_n = \dim V_n$. Hence

$$[\sum_{k=1}^{n} (-1)^k \dim V_k = \dim N_2 - (\dim N_2 - \dim N_3) + (\dim N_4 - \dim N_3) - \cdots \pm (\dim N_{n-1} - \dim N_n) \mp \dim N$$

□

We now apply this result to composition.

Proposition 5.9 *Let $S, T\colon H \to H$ be Fredholm operators. Then ST is a Fredholm operator, with $Ind(ST) = Ind(S) + Ind(T)$.*

Proof: It follows from Atkinson's theorem that ST is a Fredholm operator. To obtain the formula for the index, observe we have an exact sequence

$$0 \to \ker T \to \ker(ST) \to \ker S \to \frac{H}{im\, T} \to \frac{H}{im(ST)} \to \frac{H}{im\, S} \to 0$$

under the canonical inclusion and quotient maps. Hence by the above lemma, we have that

$$\dim(\ker T) - \dim(\ker ST) + \dim(\ker S) - \dim(H/\, im\, T) + \dim(H/\, im(ST)) - \dim(H/\, im\, S) = 0.$$

Applying proposition 5.6, we see that

$$Ind\, T - Ind(ST) + Ind\, S = 0$$

and the result follows. □

Corollary 5.10 *Let $T\colon H \to H$ be a Fredholm operator, and let $A\colon H \to H$ be a bounded linear map which is invertible. Then the operators AT and TA are Fredholm, with the same index as T.*

Proof: This follows immediately from the above, and the fact that A is a Fredholm operator with index zero. □

Recall that we define the *trace* of a matrix to be the sum of its diagonal elements. The following notion is a generalisation.

Definition 5.11 *Let H be a Hilbert space. Let $T\colon H \to H$ be a linear map with finite-dimensional image. Choose an orthonormal basis, $\{e_1, \ldots, e_n\}$ for the image $T[H]$. Then we define the* trace *of T by the formula*

$$tr(T) = \sum_{j=1}^{n} \langle e_j, Te_j \rangle.$$

The trace has the following properties; the proof is left as an exercise.

Proposition 5.12 *Let $T\colon H \to H$ be a linear map with finite-dimensional image. Then:*

- *The trace, $tr(T)$, is independent of choice of basis for the space $T[H]$.*
- *Let $S\colon H \to H$ be another linear map with finite-dimensional image. Then the maps ST and TS have finite-dimensional images, and $tr(ST) = tr(TS)$.*
- *Let $\alpha, \beta \in \mathbb{F}$. Then the operator $\alpha S + \beta T$ has finite-dimensional image, and $tr(\alpha S + \beta T) = \alpha\, tr(S) + \beta\, tr(T)$.*

- *If P is a projection with finite-dimensional image, then $tr(P) = \dim P[H]$.*

□

The notion of trace is used at the end of the following lemma.

Lemma 5.13 *Let $A: H \to H$ be a bounded linear map with finite-dimensional image. Then $A + I$ is a Fredholm operator with index 0.*

Proof: Observe

$$I(A+I) - I = A \qquad (A+I)I - I - A.$$

Since the operator A has finite-dimensional image, it is compact. Hence, by Atkinson's theorem, the operator $A + I$ is Fredholm.

Set $T = A + I$. We need to show that $Ind(A + I) = 0$. Observe that the restriction $T_0: (\ker T)^{\perp} \to T[H]$ is injective. It is clearly also surjective. So the map T_0 is bijective, and as $T[H]$ is closed, we can deduce by the open mapping theorem that there is a bounded linear map $S_0: T[H] \to (\ker T)^{\perp}$ that is the inverse of T_0.

Again, because the image $T[H]$ is closed, we can write $H = T[H] \oplus T[H]^{\perp}$. Let us extend S_0 to a bounded linear map $S: H \to H$ by setting $S(v) = 0$ whenever $v \in T[H]^{\perp}$.

Define $P = I - ST$. Let $x \in H$. Write $x = y + z$, where $y \in \ker T$ and $z \in (\ker T)^{\perp}$. Then we have $T(y) = 0$ and $ST(z) = z$, so

$$P(y) = (I - ST)y = y$$

and

$$P(z) = (I - ST)z = z - (ST)z = z - z = 0.$$

Thus $P(y + z) = y$, and it follows that P is the projection onto $\ker T$. Similarly, if we define $Q = I - TS$, then Q is the projection onto $\ker T^*$. Hence

$$Ind(T) = \dim P[H] - \dim Q[H].$$

Note that $P - Q = TS - ST = AS - SA$. Define a subspace

$$H' = P[H] + Q[H] + A[H] + A^*[H] = \ker T + \ker T^* + A[H] + A^*[H].$$

Since the operator T is Fredholm, and the operator A has finite-dimensional image, the above is a sum of finite-dimensional spaces, and is therefore closed.

Let $R: H \to H'$ be the projection onto H'. Since P is the projection onto $\ker T \subseteq H'$, we have $PR = RP = P$. Similarly, $QR = RQ = Q$.

By definition of the subspace H', we have $RA = A$ and $RA^* = A^*$. Taking adjoints, we see $AR = A$, since R, as a projection, is self-adjoint.

Set $S_1 = RSR$. Then, using these identities

$$\begin{aligned} AS_1 - S_1A &= ARSR - RSRA \\ &= ASR - RSA \\ &= RASR - RSAR \\ &= R(AS - SA)R \\ &= R(P - Q)R \\ &= P - Q \end{aligned}$$

Now, since the operators P and Q are projections, by looking at the trace we have

$$Ind(T) = \dim P[H] - \dim Q[H] = tr(P - Q) = tr(AS_1 - S_1A) = 0.$$

This completes the proof. □

Theorem 5.14 *Let H be a Hilbert space, and let $Fred(H)$ be the set of all Fredholm operators on H. Then the map $Ind\colon Fred(H) \to \mathbb{Z}$ defined by taking the index of a Fredholm operator is continuous.*

Proof: Let $P\colon H \to H$ be the projection onto $\ker T$. By the Hilbert space projection theorem, we can find $R\colon H \to H$ such that $RT = I - P$. Now

$$R(T + S) = RT + RS = I - P + RS$$

By the Carl Neumann criterion, if $\|S\| < \|R\|^{-1}$, then the operator $I + RS$ is invertible. By definition, P is a finite rank operator, so $I - P$ is Fredholm, and $Ind(I - P) = 0$. Hence RT is Fredholm, and $Ind(RT) = 0$. By the above result, we see that R is Fredholm, and $Ind\, R = -\,Ind\, T$. Similarly

$$Ind(R(T + S)) = Ind(I - P + RS) = 0$$

since P is finite rank and $I + RS$ is invertible. We conclude $Ind\, R = -\,Ind(T + S)$, and therefore $Ind(T) = Ind(T + S)$ as required. □

Corollary 5.15 *Let $T\colon [0, 1] \to Fred(H)$ be a continuous path of Fredholm operators. Then $Ind\, T(0) = Ind\, T(1)$.*

Proof: The map $Ind \circ T\colon [0, 1] \to \mathbb{Z}$ is continuous. If $Ind\, T(0) \neq Ind\, T(1)$, then by the intermediate value theorem, we have $t \in [0, 1]$ such that $Ind\, T(t) \notin \mathbb{Z}$. But this is impossible, as the index of a Fredholm operator is an integer. □

Our final major property of the Fredholm index is termed *stability*.

Theorem 5.16 *Let $T\colon H \to H$ be a Fredholm operator, and let $K\colon H \to H$ be compact. Then $T + K$ is Fredholm, and $Ind(T + K) = Ind(T)$.*

Proof: By Atkinson's theorem, we have operators $S_1, S_2 \colon H \to H$ and compact operators K_1 and K_2 such that $S_1 T = I + K_1$ and $TS_2 = I + K_2$. We can check that

$$S_1(T+K) = I + K_1' \qquad (T+K)S_2 = I + K_2'$$

where K_1' and K_2' are compact. Again applying Atkinson's theorem, $T + K$ is a compact operator.

Now any compact operator is a norm limit of finite rank operators, and for any finite rank operator F, we know that $Ind(I + F) = 0$. Hence, by theorem 5.14, $Ind(I + K_1) = 0$. By multiplicity of the Fredholm index

$$0 = Ind(I + K_1) = Ind(S_1 T) = Ind\, S_1 + Ind\, T$$

So $Ind(S_1) = -Ind(T)$. Similarly

$$0 = Ind(I+K_1') = Ind(S_1(T+K)) = Ind(S_1)+Ind(T+K) = -Ind(T)+Ind(T+K).$$

The result now follows. □

Corollary 5.17 *Let $T \colon H \to H$ be a Fredholm operator. Let S be a parametrix for T (ie: an inverse, modulo compacts, which exists by Atkinson's theorem). Then S is also Fredholm, and $Ind(S) = -Ind(T)$.*

Proof: The fact that S is Fredholm is immediate from Atkinson's theorem. Let $K = ST - I$. Then K is compact, and by the above theorem and stability, we have

$$0 = Ind(I + K) = Ind(ST) = Ind(S) + Ind(T)$$

so the result follows. □

5.3 The Toeplitz Index

Consider the unit circle in $\mathbb{C}$

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$

Let $C(\mathbb{T})$ be the complex vector space of continuous maps $\mathbb{T} \to \mathbb{C}$. We have an inner product on $C(\mathbb{T})$ defined by the formula

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^1 \overline{f(e^{it})} g(e^{it})\, dt.$$

We denote the completion of $C(\mathbb{T})$ with respect to this inner product by $L^2(\mathbb{T})$. The following is left as an exercise.

Proposition 5.18 *Let $k \in \mathbb{Z}$. Define $e_k \in C(\mathbb{T})$ by the formula $e_k(z) = z^k$. Then the set $\{e_k \mid k \in \mathbb{Z}\}$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{T})$.* □

Definition 5.19 *We define the* Hardy space*, $H^2(\mathbb{T})$, to be the closed subspace of $C(\mathbb{T})$ with orthonormal basis $\{e_n \mid n \geq 0\}$.*

Now, let $f \in C(\mathbb{T})$. For any continuous map $g \in C(\mathbb{T})$, we can define a new continuous map $M_f(g) \in C(\mathbb{T})$ by the formula $M_f(g)(z) = f(z)g(z)$. Let $\|f\|_\infty = \sup\{|f(z)| \mid z \in \mathbb{T}\}$.

Observe

$$\|M_f(g)\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})g(e^{it})|^2 \, dt \leq \frac{\|f\|_\infty^2}{2\pi}\int_0^{2\pi} |g(e^{it})|^2 \, dt = \|f\|_\infty^2 \|g\|^2.$$

Hence the map $M_f\colon C(\mathbb{T}) \to C(\mathbb{T})$ extends to a bounded linear map $M_f\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$. Moreover, $\|M_f\| \leq \|f\|_\infty$.

Definition 5.20 *Let $f \in C(\mathbb{T})$. We call the map $M_f\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ the* multiplication operator *associated to f.*

Let $P\colon L^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the orthogonal projection onto the subspace $H^2(\mathbb{T})$. Then we define the Toeplitz operator *associated to f to be the map $T_f\colon H^2(\mathbb{T}) \to H^2(\mathbb{T})$ defined by the formula $T_f(g) = P \circ M_f(g)$ where $g \in H^2(\mathbb{T})$.*

The assignment $f \mapsto T_f$ defines a linear map $C(\mathbb{T}) \to \mathcal{L}(H^2(\mathbb{T}))$. We saw above that $\|M_f\| \leq \|f\|_\infty$. Hence $\|T_f\| \leq \|f\|_\infty$, meaning that the linear map $f \mapsto T_f$ is continuous.

A remarkable result called the *Toeplitz index theorem* tells us that if f is never zero, then the Toeplitz operator T_f is Fredholm, and that

$$Ind(T_f) = -Wind(f; 0).$$

Thus analysis, in the form of the Fredholm index of the Toeplitz operator, is intimately connected to geometry, in the form of the winding number. Our aim in the remainder of this chapter is to prove the Toeplitz index theorem.

Our first step is to show that the operator T_f is Fredholm.

Lemma 5.21 *Let $f, g \in C(\mathbb{T})$. Then the operator $T_f T_g = T_{fg}$ is compact.*

Proof: Let $g \in C(\mathbb{T})$ and $h \in H^2(\mathbb{T})$. Write

$$g = \sum_{k=-\infty}^{\infty} \alpha_k e_k \qquad h = \sum_{n=0}^{\infty} \beta_n e_n.$$

Observe that

$$T_g h = P\left(\sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty} \alpha_k \beta_n e_{k+n}\right) = \sum_{n=0}^{\infty}\sum_{k=-n}^{\infty} \alpha_k \beta_n e_{k+n}.$$

Let $m \in \mathbb{Z}$. Observe

$$T_{e_m} T_g h = P\left(\sum_{n=0}^{\infty}\sum_{k=-n}^{\infty} \alpha_k \beta_n e_{k+n+m}\right) = \sum_{n=0}^{\infty}\sum_{k=\max(-n,-m-n)}^{\infty} \alpha_k \beta_n e_{k+n+m}.$$

On the other hand

$$T_{e_m g}h = P\left(\sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty}\alpha_k\beta_n e_{k+n+m}\right) = \sum_{n=0}^{\infty}\sum_{k=-m-n}^{\infty}\alpha_k\beta_n e_{k+n+m}.$$

Thus if $n < 0$, then $T_{e_n g} = T_{e_n}T_g$, so $T_{e_n}T_g - T_{e_n g} = 0$, which is certainly compact.

On the other hand, if $n \geq 0$, then for any $h \in H^2(\mathbb{T})$, we have

$$(T_{e_n}T_g - T_{e_n g})(h) \in \mathrm{Span}\{e_0, \ldots, e_{n-1}\}$$

so $T_{e_n}T_g - T_{e_n g}$ has finite-dimensional image, and is therefore compact.

Now, note that the map $f \mapsto T_f$ is linear. Let $p \in C(\mathbb{T})$ be a Laurent polynomial, that is to say

$$p(z) = \sum_{k=-m}^{n} a_k z^k.$$

for some $m, n \in \mathbb{N}$ and $a_k \in \mathbb{C}$. Then by the above, the map $T_pT_g - T_{pg}$ is a finite linear combination of compact operators, and so itself compact.

Let $f \in C(\mathbb{T})$. Then by the Stone-Weierstrass theorem, there is a sequence of Laurent polynomials, p_n such that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$; the precise details of the argument are left as an exercise. Therefore, by continuity of the formation of the Toeplitz operator, $\|T_{p_n} - T_f\| \to 0$ as $n \to \infty$. It follows that (in the operator norm)

$$T_{p_n}T_g - T_{p_n g} \to T_fT_g - T_{fg}$$

as $n \to \infty$.

Thus the operator $T_fT_g - T_{fg}$ is a limit of compact operators, and so itself compact. □

Theorem 5.22 *Let $f \in C(\mathbb{T})$ be such that $f(z) \neq 0$ for all $z \in \mathbb{T}$. Then the Toeplitz operator T_f is Fredholm.*

Proof: By the above, we can check that the operator T_f has parametrix $T_{1/f}$; the precise details are left as an exercise. □

The following lemma is also left as an exercise.

Lemma 5.23 *Let $f \in C(\mathbb{T})$. Then $T_f^* = T_{\overline{f}}$.*

We now look at the Fredholm index of Toeplitz operators. We begin with a computation.

Proposition 5.24 *Let $k \in \mathbb{Z}$. Let $e_k(z) = z^n$. Then $Ind(T_{e_k}) = -k$.*

Proof: Let $k \geq 0$. Let $g = \sum_{n=0}^{\infty} \alpha_n e_n \in H^2(\mathbb{T})$. Then

$$T_{e_k} g = \sum_{n=0}^{\infty} \alpha_n e_{n+k}.$$

For $z \in \mathbb{T}$, we have $1/z = \overline{z}$; hence, by lemma 5.23, $e_k^* = e_{-k}$. Now

$$T_{e_{-k}} g = \sum_{n=k}^{\infty} \alpha_n e_{n-k}.$$

Thus $\dim \ker(T_{e_k}) = 0$, and $\dim \ker(T_{e_k}^* = k$. Therefore $Ind(T_{e_k}) = -k$. The result when $k < 0$ also easily follows from the above calculations. □

Proposition 5.25 *Let $f, g \in C(\mathbb{T})\backslash\{0\}$. Suppose we have a path from f to g in the space $C(\mathbb{T})\backslash\{0\}$. Then $Ind(T_f) = Ind(T_g)$.*

Proof: Let $\gamma\colon [0,1] \to C(\mathbb{T})$ be a path from f to g, that is to say γ is continuous, $\gamma(0) = f$, and $\gamma(1) = g$. Then we have a continuous map $T_\gamma\colon [0,1] \to Fred\, H^2(\mathbb{T})$ defined by the formula $T_\gamma(t) = T_{\gamma(t)}$.

Now, the Fredholm index defines a continuous map $Ind\colon Fred\, H^2(\mathbb{T}) \to \mathbb{Z}$. So we have a continuous map $Ind \circ T_\gamma\colon C(\mathbb{T})\backslash\{0\} \to \mathbb{Z}$ such that $Ind \circ T_\gamma(0) = Ind(T_f)$ and $Ind \circ T_\gamma(1) = Ind(T_g)$.

Suppose $Ind(T_f) \neq Ind(T_g)$. Say $Ind(T_f) < Ind(T_g)$. Then, as $Ind(T_f)$ and $Ind(T_g)$ are integers, by the intermediate value theorem, we would have $t \in [0,1]$ such that $Ind(T_{\gamma(t)}) = Ind(T_f) + \frac{1}{2}$. But $Ind(T_{\gamma(t)}) \in \mathbb{Z}$, so this statement is a contradiction.

We similarly run into a contradiction if $Ind(T_f) > Ind(T_g)$. Thus $Ind(T_f) = Ind(T_g)$ and we are done. □

Now in order to prove the Toeplitz index theorem, we need some computations involving winding numbers.

To be concrete, given a continuous map $f\colon \mathbb{T} \to \mathbb{C}\backslash\{0\}$, we can form the winding number $Wind(f;0)$; this number is the winding number of the path $\alpha\colon [0,2\pi] \to \mathbb{C}\backslash\{0\}$ defined by the formula $\alpha(s) = f(e^{is})$, that is to say

$$Wind(f;0) = \frac{1}{2\pi i} \oint_\alpha \frac{1}{z}\, dz.$$

Proposition 5.26 *We have $Wind(e_k;0) = k$.*

Proof: Let $\alpha_k(s) = e_k(e^{is}) = (e^{is})^k = e^{iks}$. Then we have

$$Wind(e_k;0) = \frac{1}{2\pi i} \oint_{\alpha_k} \frac{1}{z}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ike^{iks}}{e^{iks}}\, ds = k.$$

□

Proposition 5.27 *Let $f, g \in C(\mathbb{T})\backslash\{0\}$. Suppose we have a path from f to g in the space $C(\mathbb{T})\backslash\{0\}$. Then $Wind(f;0) = Wind(g;0)$.*

Proof: Define $\alpha, \beta\colon [0, 2\pi] \to \mathbb{C}\backslash\{0\}$ by the formula $\alpha(s) = f(e^{is})$ and $\beta(s) = g(e^{is})$ respectively. Let γ be a path from f to g in the space $C(\mathbb{T})\backslash\{0\}$ as stated.

Note that $\gamma(t) \in C(\mathbb{T})\backslash\{0\}$ for all $t \in [0,1]$. Observe that we can define a homotopy $H\colon [0, 2\pi] \times [0,1] \to \mathbb{C}\backslash\{0\}$ from α to β by the formula

$$H(s,t) = \gamma(t)(s) \qquad s \in [0, 2\pi],\ t \in [0,1].$$

Hence, by homotopy invariance of the winding number

$$Wind(f;0) = Wind(\alpha;0) = Wind(\beta;0) = Wind(g;0)$$

and we are done. □

Now, the computation of the fundamental group of the circle (and by extension the space $\mathbb{C}\backslash\{0\}$) in algebraic topology gives us the following (non-trivial) result, which is the last ingredient we need.

Proposition 5.28 *Let $f \in C(\mathbb{T})\backslash\{0\}$. Then there is a path from f to the map e_k for some $k \in \mathbb{Z}$.* □

We are now ready for our main result.

Theorem 5.29 *Let $f \in C(\mathbb{T})\backslash\{0\}$. Then $Ind(T_f) = -Wind(f;0)$.*

Proof: By proposition 5.28, we have a path from f to e_k for some $k \in \mathbb{Z}$. By propositions 5.25 and 5.27 we have

$$Ind(T_f) = Ind(T_{e_k}) \qquad Wind(f;0) = Wind(e_k;0).$$

By propositions 5.24 and 5.26 we have

$$Ind(T_{e_k}) = -k \qquad Wind(e_k;0) = k.$$

The result now follows. □