Functional Analysis: Exercises

Paul D. Mitchener

March 2, 2018

1. Let V be a normed vector space.

   (a) What is meant by the statement that V is a Banach space ?

   (b) Let l^1 be the vector space of all sequences $(a_1, a_2, a_3, \ldots)$ of real numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Show that l^1 has a norm defined by the formula
   $$\|(a_n)\| = \sum_{n=1}^{\infty} |a_n|.$$

   (c) Show that the space $\mathbb{R}^n$ is a Banach space.

   (d) Show that l^1 is a Banach space.

2. (a) Let V and W be normed vector spaces. Say what is meant when we describe a linear map $T: V \to W$ as a *bounded linear map.*

   (b) State the open mapping theorem.

   (c) Give an example of a bounded linear map between normed vector spaces which is not open.

   (d) State the closed graph theorem.

   (e) et H be a Hilbert space. Let $f, g: H \to H$ be linear maps where
   $$\langle f(v), w \rangle = \langle v, g(w) \rangle$$
   for all $v, w \in H$. Prove that f and g are bounded linear maps.

3. Let V be a real normed vector space. We say a function $f: V \to \mathbb{R}$ is a *Lipschitz function* if there is a constant $C > 0$ such that
   $$|f(x) - f(y)| \leq C\|x - y\|$$
   for all $x, y \in V$.

   Let $\Lambda(V)$ be the set of all Lipschitz functions $f: V \to \mathbb{R}$ such that $f(0) = 0$.

   (a) Prove that $\Lambda(V)$ is a normed vector space, where for $f \in \Lambda(V)$, the norm $\|f\|$ is the smallest constant C for which $|f(x) - f(y)| \leq C\|x - y\|$ for all $x, y \in V$.

(b) Show that the dual space, V^*, is a subspace of $\Lambda(V)$.

(c) Give an example of an element of a vector space V, and an element $f \in \Lambda(V)$ that does not belong to the dual space V^*. Justify your answer.

(d) Let $f \in \Lambda(V)$, $x \in V$. Show that

$$|f(x)| \leq \|f\| \cdot \|x\|.$$

(e) Show that the space $\Lambda(V)$ is complete.

4. Let V be an inner product space.

(a) Prove that V has a norm defined by the formula $\|v\| = \sqrt{\langle v, v\rangle}$. You may use the Cauchy-Schwarz inequality without proof.

(b) Prove that the norm satisfies the *parallelogram law*: $\|u+v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$ for all $u, v \in V$.

(c) Let V be a real normed vector space, where the norm satisfies the parallelogram law. Prove that V has an inner product given by the formula

$$\langle u, v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2.$$

(d) Give an example of a real normed vector space where the norm does not arise from an inner product. Justify your answer fully.

5. (a) Let $C[0,1]$ be the real vector space of all continuous functions $[0,1] \to \mathbb{R}$.

(b) Show that we have a norm on $C[0,1]$ defined by the formula

$$\|f\| = \sup\{|f(t)| \mid t \in [0,1]\}.$$

(c) Show that $C[0,1]$ is complete with respect to this norm.

(d) State the Stone-Weierstrass theorem for real-valued functions.

(e) Use the Stone-Weierstrass theorem to show that the set of all polynomials is a dense subset of $C[0,1]$.

6. (a) State the Stone-Weierstrass theorem for complex-valued functions.

(b) Let

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Let $f_k(z) = z^k$. Show that the span of the set $\{z^k \mid k \in \mathbb{Z}\}$ is dense in $C(\mathbb{T})$.

(c) Let $C_p[0, 2\pi]$ be the Banach space of continuous functions $f\colon [0, 2\pi] \to \mathbb{C}$ such that $f(0) = f(2\pi)$, under the norm

$$\|f\|_\infty = \sup\{|f(x)| \mid x \in [0, 2\pi]\}.$$

Write down an isometric isomorphism $\alpha\colon C(\mathbb{T}) \to C_p[0, 2\pi]$.

(d) Let $e_k(t) = \exp(ikt)$. Show that $\{e_k \mid k \in \mathbb{Z}\}$ is an orthonormal basis for the space $L^2[0, 2\pi]$. You may use without proof the fact that a dense subset of $C_p[0, 2\pi]$ under the above norm is also a dense subset of $L^2[0, 2\pi]$.

(e) Let $f(x) = x$. Find coefficients $a_k \in \mathbb{C}$ such that the series

$$\sum_{k=-\infty}^{\infty} a_k e_k = f$$

in the space $L^2[0, 2\pi]$.

(f) Use the above to evaluate the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

7. Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$. Let $C(\mathbb{D})$ be the vector space of continuous functions $f\colon \mathbb{D} \to \mathbb{C}$.

(a) Prove that we have a norm on $C(\mathbb{D})$ defined by the formula

$$\|f\| = \sup\{|f(z)| \mid z \in \mathbb{D}\}$$

and that $C(\mathbb{D})$ is complete with respect to this norm.

(b) State the Stone-Weierstrass theorem for complex-valued functions.

(c) Which of the following are dense subsets of $C(\mathbb{D})$? Justify your answer fully.

- The set of all complex polynomial functions $p\colon \mathbb{D} \to \mathbb{C}$.
- The set of all complex polynomial functions $p\colon \mathbb{D} \to \mathbb{C}$ of even degree.
- The set of all continuous functions $f\colon \mathbb{D} \to \mathbb{C}$ that extend to holomorphic functions in an open set containing $\mathbb{D}$.

8. (a) Let V and W be normed vector spaces. Prove that if T is a bounded linear map, then the kernel, $\ker T$, is closed.

(b) Let $C^\infty[0, 1]$ be the vector space of all real-valued infinitely-differentiable functions $f\colon [0, 1] \to \mathbb{R}$, equipped with the norm

$$\|f\| = \sup\{|f(t)| \mid t \in [0, 1]\}.$$

Let $C_0^\infty[0, 1]$ be the subspace of all infinitely differentiable functions $f\colon [0, 1] \to \mathbb{R}$ such that $f(0) = 0$.

(c) Prove that we have a bounded linear map $T\colon C^\infty[0, 1] \to C_0^\infty[0, 1]$ defined by the formula

$$(Tf)(x) = \int_0^x f(t)\, dt.$$

(d) Prove that the above map T is bijective.

(e) Does the map T have a *continuous* inverse ? Justify your answer.

(f) Say what is meant by an open map, and state the open mapping theorem.

(g) Prove that $C_0^\infty[0,1]$ is a closed subspace of the space $C^\infty[0,1]$?

(h) Is the normed vector space $C^\infty[0,1]$ complete? Justify your answer.

9. (a) Let V be a normed vector space over the field $\mathbb{R}$. Say what is meant by a bounded linear map $f\colon V \to \mathbb{R}$, and define the norm of such a map. Prove that this norm satisfies the axioms required to turn the dual space $V^* = Hom(V, \mathbb{R})$ into a normed vector space.

(b) Prove that the above dual space V^* is complete.

(c) State the Hahn-Banach theorem.

(d) Prove that the linear map $\tau\colon V \to (V^*)^*$ defined by the formula

$$\tau(v)(f) = f(v) \qquad f \in V^*,\ v \in V$$

is an isometry.

10. (a) State Zorn's lemma, including definition of the terms *maximal* and *upper bound.*

(b) Define what is meant by an orthonormal basis for a Hilbert space H.

(c) Prove using Zorn's lemma that every Hilbert space has an orthonormal basis.

11. (a) Say what is meant by an orthonormal sequence in a Hilbert space, and what is meant by saying that an orthonormal sequence is an orthonormal basis.

(b) Let H be a Hilbert space, and let (v_n) be an orthonormal sequence in H. Let $v \in H$. Suppose that

$$\|v\|^2 = \sum_{n=0}^{\infty} |\langle e_n, v\rangle|^2.$$

Prove that $v \in \overline{Span(v_n)}$.

[Hint: Look at the norm of $w_N = v - \sum_{n=1}^{N} \langle e_n, v\rangle e_n$.]

(c) Let

$$v_0 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{4}}, -\frac{1}{\sqrt{8}}, -\frac{1}{\sqrt{16}}, \ldots\right).$$

Define a bounded lineear operator $S\colon l^2 \to l^2$ be the formula $S(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots)$. Let $v_n = S^n v_0$.

Let (e_n) be the standard basis for the space l^2. Compute the sum

$$\sum_{n=0}^{\infty} |\langle e_n, v_k\rangle|^2.$$

(d) Show that the sequence (v_n) is an orthonormal basis for l^2. You may use any standard theorems from the course.

12. (a) Let H be a Hilbert space. Let $w \in H$. Show that

$$\|w\| = \sup\{|\langle v, w\rangle| \mid v \in H,\ \|v\| \leq 1\}.$$

You may use the Cauchy-Schwarz inequality without proof.

(b) State the *Riesz representation theorem* for bounded linear functionals on a Hilbert space.

(c) Let $T\colon H \to H$ be a bounded linear map. Prove that there is a unique bounded linear map $T^*\colon H \to H$ such that $\langle T^*v, w\rangle = \langle v, Tw\rangle$ for all $v, w \in H$.
Show further that $\|T^*\| = \|T\|$.

(d) Let $T\colon H \to H$ be a bounded linear map. Show that

$$T[H]^{\perp} = \ker T^*.$$

(e) Define bounded linear maps $A, B\colon l^2 \to l^2$ by the formulae

$$A(a_1, a_2, a_3, a_4 \ldots) = (0, a_1, \frac{a_2}{2}, \frac{a_3}{3}, \frac{a_4}{4}, \ldots)$$

and

$$B(a_1, a_2, a_3, a_4 \ldots) = (a_2, \frac{a_1}{2}, a_4, \frac{a_3}{2}, a_6, \frac{a_5}{2}, \ldots)$$

respectively. Compute A^* and B^*.

13. (a) Let $f, g \in L^1(\mathbb{R})$ be continuous functions. Show that we have a well-defined function $f * g \in L^1(\mathbb{R})$ defined by the formula

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

(b) For continuous functions $f, g \in L^1(\mathbb{R})$, show that, when taking the Fourier transform, $\widehat{f * g}(\omega) = \hat{f}(\omega)\hat{g}(\omega)$.

(c) Let $\alpha > 0$. Let $f \in L^1(\mathbb{R})$ be continuous. Let $g(x) = f(\alpha x)$. Show that, when considering the Fourier transform, we have $\hat{g}(\omega) = \frac{1}{\alpha}\hat{f}(\frac{\omega}{\alpha})$.

(d) Let $\sigma > 0$. Let

$$g_\sigma = \frac{1}{\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right).$$

(e) Calculate the Fourier transform $\hat{g}_\sigma$.

(f) Show that for any $\sigma, \tau > 0$, we have $g_\sigma * g_\tau = g_{\sqrt{\sigma^2+\tau^2}}$.
You may assume here, if desired, the Fourier inversion theorem.

14. For this question, you may use without proof the fact that if $\psi \in L^2(\mathbb{R})$ and $\phi \in L^1(\mathbb{R})$ then $\psi * \phi \in L^2(\mathbb{R})$, the convolution formula for the Fourier transform, and the fact that the Fourier transform of a piecewise-continuous integrable function is bounded.

    (a) Let $f, g \in L^1(\mathbb{R})$ be continuous functions. Prove that we have a well-defined integrable function $f * g$ defined by the formula

    $$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t)g(t)\ dt.$$

    (b) Show that $(f * g) * h = f * (g * h)$.

    (c) Show that $L^1(\mathbb{R})$ is a Banach algebra, with multiplication defined by taking the convolution.

    (d) State the definition of an admissible wavelet.

    (e) Let ψ be an admissible wavelet, and let $\phi \in L^1(\mathbb{R})$. Show that the convolution $\psi * \phi$ is an admissible wavelet.

15. (a) Let H be a Hilbert space. State what is meant by the statements that a map $T\colon H \to H$ is a *self-adjoint operator*, and that a map $U\colon H \to H$ is a *unitary operator*.

    (b) Let $A\colon H \to H$ be a bounded linear operator. Show that we have a well-defined operator

    $$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} \qquad (A^0 = I).$$

    (c) Let $n \in \mathbb{N}^{\geq 0}$. Show that $(e^A)^n = e^{nA}$.

    (d) Show that the operator e^A is invertible (even if A is not invertible), and $(e^A)^{-1} = e^{-A}$.

    (e) Let $A\colon H \to H$ be self-adjoint. Show that e^{iA} is unitary.

16. (a) Let A be a unital Banach algebra, and let $x \in A$ be an element such that $\|x\| < 1$. Prove that the element $1 - x$ is invertible.

    (b) Define the *spectrum* of an element $x \in A$. Prove this it is a bounded subset of $\mathbb{C}$.

    (c) State the spectral mapping theorem for polynomials.

    (d) Let A be the Banach algebra of all bounded linear maps $\ell^1 \to \ell^1$. Determine the spectrum of the following operators from ℓ^1 to ℓ^1.

    - $S(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots)$.
    - $T(a_1, a_2, a_3, \ldots) = (a_3 - 2a_1, a_4 - 2a_2, a_5 - 2a_3, \ldots)$.
      [Hint: For part (a), find the eigenvalues of S. You may use without proof the fact that the spectrum of S is closed.]

17. (a) Let A be a unital Banach algebra. Define the *spectrum* of an element $x \in A$. Prove that it is a closed subset of $\mathbb{C}$.

(b) State the spectral mapping theorem for polynomials.

(c) Define maps $S, T\colon L^2[0,\infty) \to L^2[0,\infty)$ by the formulae

$$(Sf)(x) = f(x+1) \qquad (Tf)(x) = f(x) + f(x+2).$$

Show that S and T are bounded linear maps. Find $Spectrum(S)$ and $Spectrum(T)$.

18. (a) Define what is meant by the statement that a linear map between normed vector spaces is a *compact operator*.

(b) Let $K\colon V \to W$ be a compact operator between normed vector spaces V and W. Let (x_n) be a bounded sequence in V. Prove that (Kx_n) has a convergent subsequence.

(c) Prove that any bounded linear map with finite-dimensional image is compact.

(d) Prove that the operator $S\colon \ell^2 \to \ell^2$ defined by the formula

$$S(a_1, a_2, a_3, \ldots) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, \ldots)$$

is compact.

[You may use without proof here the fact that a norm-limit of a sequence of compact operators is again compact]

19. (a) What is the definition of a Fredholm operator?

(b) Define $T\colon \ell^2 \to \ell^2$ by the formula

$$T(a_1, a_2, a_3, \ldots) = (a_1 + a_3, \frac{a_2}{2} + a_4, \frac{a_3}{3} + a_5, \ldots)$$

Show that T is Fredholm, and calculate $Index(T)$. You may use any standard results from the theory of Fredholm operators to do this.

20. (a) State the open mapping theorem.

(b) Let H be a Hilbert space, and let $V \subseteq H$ be a subspace. Define the subspace $V^\perp$, and show that $V^\perp = \overline{V}^\perp$.

(c) State the Hilbert space decomposition theorem, and use it to prove that a subspace $V \subseteq H$ is dense if and only if $V^\perp = \{0\}$.

(d) Let $(e_n)_{n=1}^\infty$ be an orthonormal sequence in H. Suppose that

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle e_n, x\rangle|^2$$

for all $x \in H$. Prove that (e_n) is an orthonormal basis for H.

(e) Let H be a Hilbert space, and let $T\colon H \to H$ be a bounded linear map with closed image. Prove that the restriction

$$T|_{(\ker T)^\perp}\colon (\ker T)^\perp \to T[H]$$

is invertible, with inverse a bounded linear map.

(f) Show that $T[H]^\perp = \ker T^*$.

21. Throughout this question, you may use any of the results proved in the previous question.

(a) Let H be a Hilbert space. Let $V \subseteq H$ be a closed subspace, and $W \subseteq H$ a finite-dimensional subspace. Prove that $V + W$ is closed.

(b) State the definition of a compact operator on H.

(c) Show that any operator with finite-dimensional image is compact.

(d) State the definition of a Fredholm operator on H.

(e) Show that if T is a Fredholm operator, we have an operator $S\colon H \to H$ such that $ST - I$ and $TS - I$ are both compact.